ADVANCED ALGEBRA

HAWKES

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MATHEMATICAL TEXTS
FOR SCHOOLS AND COLLEGES

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ADVANCED ALGEBRA

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PREFACE

To revise a text that has enjoyed a useful and active career of nearly twenty-five years is a delicate undertaking. In the present revision of the Advanced Algebra a careful attempt has been made to preserve the qualities that have contributed to the value of the original text, and at the same time to make such changes as new conditions demand.

The general aim of the text has not been modified. The introduction presents an adequate review of the essentials of elementary algebra. New material has been added to the chapters on Determinants, Permutations and Combinations, and Fractions. The treatment of Complex Numbers and the use of the method of complete induction have been somewhat curtailed. An entirely fresh set of exercises has been prepared, perhaps somewhat less forbidding than the old collection. The mathematical significance of the subjects considered has everywhere been kept in the foreground, and in a number of chapters new material has been added to round out the treatment of the topic.

Following the recommendation of the College Entrance Examination Board, the chapter on Scales of Notation has been retained, and an adequate treatment of the various topics suggested by the Board has been provided.

To acknowledge each of the suggestions received during the past twenty years which have contributed to the revision would be impossible. I cannot fail, however, to express in this place my appreciation to all who have aided in the preparation of the exercises and criticisms of the text.

H. E. HAWKES

COLUMBIA UNIVERSITY

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CONTENTS

ALGEBRA TO QUADRATICS

CHAPTER PAGE
I. FUNDAMENTAL OPERATIONS 1
II. FACTORING 19
III. FRACTIONS 32
IV. RATIO, PROPORTION, AND VARIATION 39
V. EQUATIONS 46
VI. IRRATIONAL NUMBERS AND RADICALS 68
VII. THEORY OF INDICES 84
VIII. LOGARITHMS 89
IX. QUADRATIC EQUATIONS 115

QUADRATICS AND BEYOND

X. PROPERTIES AND GRAPHS OF QUADRATIC EQUATIONS . . . 125
XI. SIMULTANEOUS QUADRATIC EQUATIONS IN TWO VARIABLES 138
XII. MATHEMATICAL INDUCTION 154
XIII. BINOMIAL THEOREM 157
XIV. ARITHMETIC PROGRESSION 163
XV. GEOMETRIC PROGRESSION 168

ADVANCED ALGEBRA

XVI. COMPLEX NUMBERS 175
XVII. THEORY OF EQUATIONS 184
XVIII. PERMUTATIONS, COMBINATIONS, AND PROBABILITY 222
XIX. DETERMINANTS 236
XX. PARTIAL FRACTIONS 266
XXI. CONTINUED FRACTIONS 277
XXII. SCALES OF NOTATION 290

INDEX 299

ADVANCED ALGEBRA

ALGEBRA TO QUADRATICS

CHAPTER I

FUNDAMENTAL OPERATIONS

1. Integers. It is assumed that the elementary operations and the meaning of the usual symbols of algebra are familiar and do not demand detailed treatment.

Undoubtedly the first numbers with which mankind was familiar were those resulting from the process of counting. These are called the positive integers. The following paragraphs indicate how, starting with the positive integers, we are led to the other numbers commonly used in algebra, through the attempt to solve the simpler types of equations.

In the brief exposition of the formal laws of algebra contained in this chapter most of the proofs are omitted.

2. Addition. The process of adding two positive integers a and b consists in finding a number x such that

$$a + b = x.$$

For any two given positive integers a single **sum** x exists which is itself a positive integer.

Thus, to find the sum of 3 and 4, we seek x such that $3 + 4 = x$.

3. Subtraction. The process of subtracting the positive number b from the positive number a consists in finding a number x such that

$$b + x = a. \tag{1}$$

This number x is called the **difference** between a and b and is denoted as follows:

$$a - b = x,$$

a being called the **minuend** and b the **subtrahend.**

To subtract 3 from 8, we seek x such that $3 + x = 8$. Thus the difference, $x, = 8 - 3$, or 5.

If a is greater than b and both are positive integers, then a single positive integer x exists which satisfies the condition expressed by equation (1).

If a is less than b, then x is not a positive integer.

For example, if $a = 6$ and $b = 9$, then $9 + x = 6$ does not lead to a value of x which is a positive integer.

In order that the process of subtraction may be possible in this case also, we introduce negative numbers, which we symbolize by $(-a)$, $(-b)$, etc., where a and b denote positive integers. When, in the difference $a - b$, a is less than b, we define $a - b = (-(b - a))$. The processes of addition and subtraction involving negative numbers are defined as follows:

$$(-(-a)) = a.^*$$
$$(-a) + (-b) = (-(a + b)).$$
$$(-a) + b = (-(a - b)).$$
$$a + (-b) = a - b.$$
$$(-a) - (-b) = (-(a - b)).$$
$$(-a) - b = (-(a + b)).$$
$$a - (-b) = a + b.$$

It is observed that the process of addition performed with positive integers always leads to positive integers. Hence, in order always to carry out this process, we do not need to enlarge our system of numbers. But the process of subtraction performed on positive integers does not always lead to positive integers; consequently, we are obliged to enlarge the set of numbers with which we are dealing, by the addition of what are called negative integers. Having postulated their existence in order that equation (1) might always have a solution, it was necessary in the foregoing definitions to show how these new numbers might be combined with positive integers and with each other without inconsistency.

---

* The symbol for a positive integer might be written thus: $(+a)$, $(+b)$, etc., consistently with the notation for negative numbers. Since, however, no ambiguity results, we omit the + sign. Since the laws of combining the positive and negative numbers given in this and the following paragraphs remove the necessity for the parentheses in the notation for the negative number, we shall omit them where there is no likelihood of confusion.

ORAL EXERCISES

In the following oral exercises find, first the sum, and second the difference, of the two numbers, always taking the second number as the minuend:

1. $-1, 3$.
2. $2, -4$.
3. $3, 1$.
4. $5, -7$.
5. $1, 2$.
6. $-2, 3$.
7. $4, -1$.
8. $-6, 7$.
9. $8, 5$.
10. $10, -15$.
11. $12, 25$.
12. $-17, 21$.

4. Zero. If in equation (1), section 3, $a = b$, there is no positive or negative number which satisfies the equation. In order that in this case also the equation may have a number satisfying it, we introduce the number *zero*, which is symbolized by 0 and defined by the equation

$$a + 0 = a,$$

or

$$a - a = 0.$$

The processes of addition and subtraction for this new number zero are defined as follows, where k stands for either a positive or a negative number:

$$0 + k = k \pm 0 = k.$$

$$0 - k = -k.$$

$$0 \pm 0 = 0.$$

5. Multiplication. The process of multiplying a by b consists in finding a number x which satisfies the equation

$$a \cdot b = x.$$

Thus, to find the product of 6 and 3, we seek the number x such that $6 \cdot 3 = x$.

When a and b are positive integers, x is a positive integer which may be found by adding a to itself b times. When the numbers to be multiplied are negative, we make the following laws:

$$(-a) \cdot (-b) = a \cdot b.$$

$$(-a) \cdot b = a \cdot (-b) = -(a \cdot b).$$

$$0 \cdot k = k \cdot 0 = 0, \qquad (1)$$

where k is a positive or negative number or zero.

These symbolic statements include the statement of the following

Principle. *A product of numbers is zero when and only when one or more of the factors are zero.*

This most important fact, which we shall use frequently, assures us that when we have a product of several numbers, as

$$a \cdot b \cdot c \cdot d = e,$$

first, if e equals zero, it is certain that one or more of the numbers a, b, c, or d are zero; *secondly,* if one or more of the numbers a, b, c, or d are zero, then e is also zero.

The operation of subtraction necessitated the introduction of the negative integers and of zero. The operation of multiplication, however, carried out with the numbers already introduced, does not lead to anything new.

6. Division. The process of dividing k by l consists in finding a number x which satisfies the equation

$$x \cdot l = k, \qquad (1)$$

where k and l are positive or negative integers or k is 0. The case where $l = 0$ is considered in section 7.

Thus, to find the quotient of 21 and 7, we seek the number x such that $7 \cdot x = 21$.

When k occurs in the sequence of numbers

$$\cdots, -3l, -2l, -l, 0, l, 2l, 3l, \cdots,$$

x is a definite integer or 0; that is, it is a number such as we have previously considered. The number k is then a multiple of l. If k is not found in this series but is between two numbers of the series, then, in order that in this case also the process may be possible, we introduce the fraction, which we symbolize by $k \div l$ or $\frac{k}{l}$ and which is defined by the equation

$$\frac{k}{l} \cdot l = k.$$

The operations for addition, subtraction, multiplication, and division of fractions are defined as follows:

$$\frac{k}{l} \pm \frac{m}{n} = \frac{kn \pm lm}{ln}. \tag{2}$$

$$\frac{k}{l} \cdot \frac{m}{n} = \frac{km}{ln}. \tag{3}$$

$$\frac{k}{l} \div \frac{m}{n} = \frac{kn}{lm}. \tag{4}$$

Further properties of the fraction are the following:

$$1 = \frac{k}{k} = \frac{1}{1}.$$

$$\frac{k}{l} = \frac{km}{lm}, \text{ where } m \text{ is any number.} \tag{5}$$

$$\frac{-k}{l} = \frac{k}{-l} = -\frac{k}{l}. \tag{6}$$

The last two equations are expressed verbally as follows:

Both numerator and denominator of a fraction may be multiplied by any number without changing the value of the fraction.

Changing the sign of either numerator or denominator of a fraction is equivalent to changing the sign of the fraction.

The laws of signs in multiplication given in section 5 may now be assumed to hold when the letters represent fractions as well as integers.

Thus, for example, $\left[-\left(\frac{a}{b}\right)\right]\cdot\left[-\left(\frac{c}{d}\right)\right]=\frac{ac}{bd}.$

The positive or negative number k may be written in the fractional form

$$\frac{k}{1}.$$

ORAL EXERCISES

In the following oral exercises find the quotient of the first number divided by the second number:

1. 5, 20.
2. – 3, 15.
3. 2, – 7.
4. 5, – 4.
5. – 3, – 6.
6. 2, 5.
7. – 1, – 3.
8. – 4, – 9.
9. 15, – 10.
10. – 21, 6.
11. 17, – 2.
12. – 12, 5.

7. Division by zero. If in equation (1), section 6, $l = 0$, there is no positive or negative integer or fraction x which satisfies the equation, since by (1), section 5, whatever value x might have, its product with zero would be zero.

Thus division by zero is entirely excluded from algebraic processes. Before any division can safely be performed, one must be assured that the divisor cannot vanish. In the equation

$$4\cdot 0 = 2\cdot 0,$$

if we should allow division of both sides of our equation by zero, we should be led to the absurd result $4 = 2$.

8. Fundamental operations. The operations of addition, subtraction, multiplication, and division we call the four **fundamental operations.** Any numbers that can be derived from the number 1 by means of the four fundamental operations are called **rational numbers.** Such numbers comprise all positive and negative integers and such fractions as have integers for numerator and denominator or can be reduced to such numbers. Positive or negative integers are called **integral numbers.**

9. Practical demand for negative and fractional numbers. In the preceding discussion negative numbers and fractions have been introduced on account of the *mathematical* necessity for them. They were needed to make the four fundamental operations always possible. That this mathematical necessity corresponds to a practical necessity appears as soon as we attempt to apply our four operations to practical affairs. Thus if on a certain day the temperature is + 20° and the next day the mercury falls 25°, in order to express the second temperature we must subtract 25 from 20. If it is desired to express the result of dividing 7 apples into 3 equal parts, fractions are necessary. If we had not introduced negative and fractional numbers, these applications would be impossible and our mathematics would be inapplicable to these cases and also to countless other problems of everyday occurrence.

The only other operations used in algebra are the extraction of roots and the raising to powers. It will be found later that the extraction of roots requires the further enrichment of the number system in order that it may always lead to a number. The raising of a number to a power, however, does not call for anything new.

10. Laws of operation. All the numbers which we use in algebra are subject to the following laws, which we assume without further discussion:

I. Commutative law of addition. The value of the sum of two numbers does not depend on the order of summation.

Symbolically expressed,

$$a + b = b + a,$$

where a and b represent any numbers such as we have presented or shall hereafter introduce.

II. Associative law of addition. The sum of three numbers does not depend on the way in which the numbers are grouped in performing the operation of addition.

Symbolically expressed,

$$a + (b + c) = (a + b) + c = a + b + c.$$

III. Commutative law of multiplication. *The value of the product of two numbers does not depend on the order of multiplication.*

Symbolically expressed,

$$a \cdot b = b \cdot a.$$

IV. Associative law of multiplication. *The value of the product of three numbers does not depend on the way in which the numbers are grouped in the operation of multiplication.*

Symbolically expressed,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot b \cdot c.$$

V. Distributive law. *The product of a single number and the sum of two numbers is identical with the sum of the products of the first number and the other two numbers taken singly.*

Symbolically expressed,

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

All the above laws are readily seen to hold when more than three numbers are involved.

11. Integral and rational expressions. A polynomial is **integral** when it may be expressed by a succession of literal terms, no one of which contains any letter in the denominator.

Thus $4x^5 - x^3 - 2x^2 - \frac{3}{4}x + 1$ is integral.

The quotient of two integral expressions is called **rational.**

Thus $\frac{x^2 - 2x + 3}{x - 7}$ is rational.

12. Operations on polynomials. We assume that the same formal laws for the four fundamental operations enunciated in sections 2 to 6 and the laws given in section 10 hold whether the letters in the symbolic statements represent numbers or polynomials.

In fact the literal expressions which we use are in essence nothing else than numeric expressions, since the letters are merely symbols for numbers. When the letters are replaced by numbers, the literal expressions reduce to numeric expressions, for which the previous laws have been explicitly given.

13. Addition of polynomials. For performing this operation we have the following

Rule. *Write the terms with the same literal part in a column. Find the algebraic sum of the terms in each column, and write the results in succession with their proper signs.*

When the polynomials reduce to monomials the same rule is to be observed.

14. Subtraction of polynomials. For performing this operation we have the following

Rule. *Write the subtrahend under the minuend so that terms with the same literal part are in the same column.*

To each term of the minuend add the corresponding term of the subtrahend, the sign of the latter having been changed.

It is generally preferable to imagine the signs of the subtrahend changed rather than actually to write it with the changed signs.

EXERCISES

1. Add: $2a^2b^2 + a^2b - 15ab + 3b$, $a^2b^2 + 2ab^2 - 5b$, $3b + 10a^2b^2 + 4ab$, and $a^2b - 3ab^2 + 4b$.

Solution.

$$\begin{array}{rrrrr}
2a^2b^2 & +\ a^2b & -\ 15ab & & +\ 3b \\
a^2b^2 & & & +\ 2ab^2 & -\ 5b \\
10a^2b^2 & & +\ 4ab & & +\ 3b \\
 & +\ a^2b & & -\ 3ab^2 & +\ 4b \\
\hline
13a^2b^2 & +\ 2a^2b & -\ 11ab & -\ ab^2 & +\ 5b
\end{array}$$

2. From $2a^2b^3 + 3a^2b + 15ab - 4a$ subtract $a^2b^3 - 4a^2b + 25ab - 3b + 5a$.

Solution.

$$\begin{array}{rrrrr}
2a^2b^3 & +\ 3a^2b & +\ 15ab & -\ 4a & \\
a^2b^3 & -\ 4a^2b & +\ 25ab & +\ 5a & -\ 3b \\
\hline
a^2b^3 & +\ 7a^2b & -\ 10ab & -\ 9a & +\ 3b
\end{array}$$

3. Add: $2x^3 + 3x^2y + 15y^3$, $3x^2y + 4xy^2 - 8y^3$, $3y^3 - 14x^3 - 10x^2y$, and $25xy^2 + 15x^2y - 4y^3$.

4. Add: $m^2 + 3\,mn + xn$, $2\,n^2 + 15\,mn - 25\,m^2$, $13\,xn - 2\,m^2 + 5\,n^2$, and $4\,mn - 2\,xn + 3\,m^2$.

5. From $2\,r + 5\,s - 15\,rs$ subtract $2\,rs - 8\,r + 3\,s$.

6. From $3\,x^2 + 5\,xy + 2\,y^2 - 21\,x^2y$ subtract $2\,x^2 + 15\,y^2 - 3\,xy$.

7. Add: $3\,a^3 + 5\,a^2b + 15\,ab^2 - 10\,b^3$, $2\,a^2b - 3\,ab^2 + 3\,b^3$, $19\,ab^2 + 3\,a^3 - 2\,b^3$, and $3\,a^2b + ab^2 - a^3$.

8. Subtract $3\,x^2 + 15\,x + 2\,y^3$ from $21\,y^3 + 15\,y - 30\,x^2$.

9. Add: $21\,a^2 + 5\,a - 3\,b + 21\,ab + 2\,b^2 - 3$, $a^2 - b^2 + 2 - a^2b + 5\,ab$, $19\,a - ab + a^2 - 5\,b^2 + 3\,a^2b$, $a + b - a^2 + 3\,b^2$, and $2 - 10\,a^2 + 15\,a^2b - 12\,b^2$.

10. From the sum of $m^2 + mn - 3\,ml$, $mn - 15\,ml$, $21\,ml + 2\,m^2 + 3$, and $2\,ml - m^2 + 15$ subtract $25\,m^2 - 14\,ml + 21 - 5\,mn$.

11. From the sum of $a^2 + 25\,ab + b^2$ and $15\,ab - a^2$ subtract $15\,a^2 + 10\,b^2 + 8\,ab - 10$.

12. From $25\,x^2 + 19\,xy - 9\,y^2 + 13$ subtract the sum of $2\,x^2 - 15\,xy$, $2\,xy - y^2 - 2\,x^2 + 10$, and $x^2 + y^2 + xy - 5$.

13. From $a + b + c$ subtract the sum of $2\,a + 3\,c - 10\,b$, $5\,a - 13\,b + 2\,c$, $8\,a - c$, and $2\,a - 3\,c + 25\,b$.

14. From the sum of $2\,r - 5\,s + 2\,q$, $25\,s + 15\,q - 2\,r$, $25\,s - 15\,r + 2\,q$, and $r + s + 10\,q$ subtract $25\,q - 5\,s + 9\,r$.

15. Subtract the sum of $5\,a + 2\,b - 10\,c^2 + 3$, $a + c^2 - 15\,b$, $21\,c^2 + 5\,ab - 2\,ac$, and $2\,ac + 5\,c^2 - 3\,ab + a - 10$ from $108\,c^2 + 15\,ab - 2\,ac + 13\,a + 21\,b - 36$.

16. From the sum of $\frac{2}{3}\,a + \frac{9}{5}\,b + \frac{10}{4}\,c$, $2\,a + \frac{1}{2}\,c - 5\,b$, and $\frac{4}{5}\,a - \frac{9}{2}\,b$ subtract $\frac{5}{8}\,a - \frac{2}{3}\,b + 8\,c$.

17. From the sum of $\frac{15}{7}\,x^2 + \frac{25}{3}\,xy + \frac{15}{2}\,y^2 + \frac{21}{4}$ and $\frac{2}{5}\,y^2 - \frac{10}{3}\,x^2 + \frac{4}{5}\,xy$ subtract $\frac{25}{24}\,x^2 + \frac{15}{6}\,xy - \frac{11}{7}\,y^2$.

15. Parentheses. When it is desirable to consider as a single symbol an expression involving several numbers or symbols for numbers, the expression is inclosed in a parenthesis. This parenthesis may then be used in operations as if it were a single number or symbol, as in fact it is, excepting that the operations inside the parenthesis may not yet have been performed and the result simplified.

Rule. *When a single parenthesis is preceded by a + sign the parenthesis may be removed, the various terms retaining the same signs.*

When a single parenthesis is preceded by a – sign the parenthesis may be removed, providing we change the signs of all the terms inside the parenthesis.

When several parentheses occur in an expression we have the following

Rule. *Remove the innermost parenthesis, changing the signs of the terms inside if the sign preceding it is minus.*

Simplify, if possible, the expression inside the new innermost parenthesis.

Repeat the process until all the parentheses are removed.

It is in general unwise to shorten the process by carrying out some of the steps in one's head. The liability to error in such attempts more than offsets the gain in time.

EXERCISES

Remove parentheses from the following:

1. $x - (2y + x) - [3x + 5y - (7y + 5x) - 3y]$.

Solution.

$$\begin{aligned} &x - (2y + x) - [3x + 5y - (7y + 5x) - 3y] \\ &= x - 2y - x - [3x + 5y - 7y - 5x - 3y] \\ &= -2y - 3x - 5y + 7y + 5x + 3y \\ &= 3y + 2x. \end{aligned}$$

2. $a - \{-b - [-(-a)]\}$.

3. $-2m + n - [n + 5m + (2m + 3n)]$.

4. $x^2 - 2xy + [-5y^2 + 10xy + (2x^2 - 3xy + 3y^2) - x^2]$.

5. $\frac{1}{2} - [\frac{5}{4} \cdot \frac{1}{3} + \frac{2}{5} - (\frac{5}{10} \cdot \frac{1}{5} - \frac{2}{3})]$.

6. $5x\{3y - 2x[5xy + 10x - 2(3x + y)]\} + 10x^2$.

7. Find the value of $a - 10b + 2a + [5a - 3c(21b - 2c) + 10a]$ when $a = 2$, $b = 3$, and $c = 1$.

8. Find the value of $m^2 + 2mn\{3n - 7m[4m - 10n(2m + n)]\}$ when $m = 1$ and $n = 5$.

9. Find the value of $-p\{r+5t-[2p+5t(21r-15t)+\frac{1}{2}t]\}$ when $p=\frac{1}{2}$, $r=2$, and $t=\frac{7}{2}$.

10. Find the value of $x^2\{-5m+2x[5x-2(3m+b)]\}$ when $x=2$, $m=5$, and $b=1$.

16. Multiplication. It is customary to write $a\cdot a=a^2$; $a\cdot a\cdot a=a^3$; $\underbrace{a\cdot a\cdots a}_{n\text{ terms}}=a^n$. We have then by the associative law of multiplication, section 10,

$$a^2\cdot a^3=(a\cdot a)(a\cdot a\cdot a)=a\cdot a\cdot a\cdot a\cdot a=a^5,$$

or, in general,

$$a^m a^n=a^{m+n}, \qquad \text{(I)}$$

where m and n are positive integers.

Furthermore,

$$(a^n)^m=\underbrace{a^n\cdot a^n\cdots a^n}_{m\text{ terms}}=a^{n\cdot m}. \qquad \text{(II)}$$

Finally,

$$a^n\cdot b^n=(a\cdot b)^n. \qquad \text{(III)}$$

The distinction between $(a^n)^m$ and a^{n^m} should be noted carefully. Thus $(2^3)^2=8^2=64$, while $2^{3^2}=2^9=512$. That is, in the expression a^{n^m} it is assumed that m is the exponent of n, not of a^n.

Equation (I) asserts that the exponent in the product of two powers of any expression is the sum of the exponents of the factors. Hence we may multiply monomials as follows:

Rule. *Write the product of the numeric coefficients, followed by all the letters that occur in the multiplier and multiplicand, each having as its exponent the sum of the exponents of that letter in the multiplier and multiplicand.*

EXAMPLE

$$-3m^5n^7p^2r^3\cdot(2mn^7p^4)=-6m^6n^{14}p^6r^3.$$

QUERY. Which of the laws in section 10 are used in the preceding example?

For the multiplication of monomials by polynomials we have the following

Rule. *Multiply each term of the polynomial by the monomial and write the resulting terms in succession.*

EXAMPLE

$$\begin{array}{r}
4xy^2 + 3x^2 + 15xy - 10y^2 + 3 \\
\underline{\qquad\qquad\qquad\qquad\qquad\qquad 2x^2y} \\
8x^3y^3 + 6x^4y + 30x^3y^2 - 20x^2y^3 + 6x^2y
\end{array}$$

QUERY. Which of the laws in section 10 are used in the preceding example?

17. Multiplication of polynomials. If in the expression for the distributive law,

$$a(c + d) = ac + ad,$$

we replace a by $x + y$, we have

$$(x + y)(c + d) = xc + yc + xd + yd,$$

which affords the

Rule. *Multiply the multiplicand by each term of the multiplier in turn, and write the partial products in succession, each preceded by the proper sign.*

To test the accuracy of the result, assume some convenient numeric value for each letter, and find the corresponding numeric value of multiplier, multiplicand, and product. The latter should be the product of the two former.

EXERCISES

Multiply and check the following:

1. $3x^2y + 2xy - 2xy^2 + y^3$ and $x + y - xy$.

Solution. **Check.** Let $x = y = 1$.

$$\begin{array}{ll}
3x^2y + 2xy - 2xy^2 + y^3 & = 4. \\
\underline{\qquad\quad x + \quad y - xy} & = 1. \\
3x^3y + 2x^2y - 2x^2y^2 + xy^3 & \\
\qquad + 3x^2y^2 - 2xy^3 + 2xy^2 + y^4 & \\
\underline{\qquad - 2x^2y^2 \qquad\qquad\qquad\qquad - 3x^3y^2 + 2x^2y^3 - xy^4} & \\
3x^3y + 2x^2y - x^2y^2 - xy^3 + 2xy^2 + y^4 - 3x^3y^2 + 2x^2y^3 - xy^4 = 4. &
\end{array}$$

2. $3a^2by$ and $2ab^3y^2$.

3. $7x^2y^3z$ and $-5x^2yz^5$.

4. $-3m^2np$ and $\dfrac{2mn}{3}$.

5. $-8a^2b^3c$ and $-7ab^4c^6$.

6. $\dfrac{15ax^2z}{2}$ and $20a^3xz$.

7. $\dfrac{12ar^2t^5}{5}$ and $-\frac{1}{2}a^2rt^3$.

8. $2\,m^2n + 3\,mp^2$ and $3\,n^2pm^4$.

9. $\dfrac{2\,ab^3c^5x^2}{3} - 4\,a^2bx^4$ and $6\,a^2bc^3x^5$.

10. $\frac{1}{2}\,x^2yz^3p^5 - \frac{2}{3}\,xy^3z^2$ and $-2\,x^2yz^4p^2$.

11. $2\,x^ab^{2b} + 3\,x^{2a}b^b$ and $4\,x^ab^b$.

12. $3\,a^2b + 2\,ab - 4\,b^2$ and $2\,a^2 - 5\,b^2 + 3\,ab$.

13. $3\,m^xn^{2y} - 2\,m^{2y} + 5\,m^2$ and $2\,m^y - 5\,mn^x$.

14. $2\,a^{m-n} + 3\,a^mb^{m+2} - a^2b^{2n-5}$ and $2\,ab^{n-3} + 4\,b^{m-2}$.

15. $8\,p^{2-a}r^{b+c}s^a - \frac{1}{2}\,r^{2+c}p^{b-a} + 3\,s^{a+b-c}p^{2ab}$ and $3\,rs^{2bc-a}$.

Expand and simplify:

16. $(2\,a + b)^3$.

17. $(m + n - p)^2 + (n + p)(p - m) - (p + 2\,n)(3\,m + n) + 2\,m(3\,p + 2\,n)$.

18. Types of multiplication. The following types of multiplication should be so familiar as merely to require inspection of the factors in order to write the product.

Rule. The product of the sum and the difference of two terms is equal to the square of the terms with like signs minus the square of the terms which have unlike signs.

EXAMPLES

$$(a - b)(a + b) = a^2 - b^2.$$
$$(3\,m + 2\,n^2)(3\,m - 2\,n^2) = 9\,m^2 - 4\,n^4.$$

19. The square of a binomial. This process is performed according to the following

Rule. The square of a binomial, or expression in two terms, is equal to the sum of the squares of the two terms plus twice their product.

EXAMPLES

$$(x + y)^2 = x^2 + y^2 + 2\,xy.$$
$$(4\,a - 2\,b)^2 = 16\,a^2 + 4\,b^2 - 16\,ab.$$

EXERCISES

Perform the following operations by inspection:

1. $(x^2 + y)^2$.
2. $(a^2 + b^3)^2$.
3. $(3\,a^r + 1)^2$.
4. $(3\,m^2 - 4\,b^2)^2$.
5. $(2\,a^{r-1} - 1)^2$.
6. $(3 + 2\,mp^2)^2$.
7. $(2\,m^2 + 4\,m + 1)^2$.
8. $(1 - 3\,x^2y^3)^2$.
9. $(m^2 + 2\,mn + n^2)^2$.
10. $(x^3 + 3\,x - 4)^2$.
11. $(x^2 - 1)(x^2 + 5\,x - 2)$.
12. $(4\,pr^2 - 5\,p^2r + 2\,pr)^2$.
13. $(2\,x^a + 3\,y^b)^2$.
14. $(2\,m^{x-2} - 3\,n^{x-1})^2$.
15. $(2\,x^m + y^n)(x^{m-1} - y^{n+2})$.
16. $(3\,x^2y + 2\,xy^2 + 3)^2$.
17. $(2\,mn - 5\,n^2)(3\,m^2 + 2\,mn)$.
18. $(-3\,x^2y + \frac{1}{2}\,xy - \frac{2}{3}\,y^2)(2\,xy - \frac{1}{2}\,y^2)$.
19. $\left(\frac{2\,a}{3} + b\right)^2$.
20. $\left(x + \frac{3}{y}\right)^2$.
21. $(2\,xy + 3\,y - \frac{3}{4}\,y^2)(3\,y - 5\,x^2)$.

20. Division. By the definition of division in section 6, we have

$$a = \frac{a^2}{a}, \quad a^3 = \frac{a^5}{a^2}, \quad a^4 = \frac{a^9}{a^5};$$

or, in general,

$$a^{n-m} = \frac{a^n}{a^m}, \tag{I}$$

where n and m are positive integers and $n > m$.

If $n = m$, we preserve the same principle and write

$$a^{n-n} = a^0 = \frac{a^n}{a^n} = 1.$$

The case where $n < m$ will be considered in Chapter VII.

For the division of monomials we have the following

Rule. *Divide the numeric coefficient of the dividend by that of the divisor for the numeric coefficient of the quotient, keeping in mind the rule of signs for division.*

Write the literal part of the dividend over that of the divisor in the form of a fraction, and perform on each pair of letters occurring in both numerator and denominator the process of division as defined by equation (I) *above.*

EXAMPLE

Divide $12\,a^4b^{11}c^2d$ by $-\,6\,a^9bc^2d^8$.

Solution. $$\frac{12\,a^4b^{11}c^2d}{-\,6\,a^9bc^2d^8} = -\frac{2\,b^{10}}{a^5d^7}.$$

The division of a polynomial by a monomial is performed according to the following

Rule. *Divide each term of the polynomial by the monomial and write the partial quotients in succession.*

EXAMPLE

Divide $8\,a^2b^6 - 12\,a^6b^2$ by $2\,a^3b^3$.

Solution. $$\frac{8\,a^2b^6}{2\,a^3b^3} - \frac{12\,a^6b^2}{2\,a^3b^3} = \frac{4\,b^3}{a} - \frac{6\,a^3}{b}.$$

For the division of a polynomial by a polynomial we have the following

Rule. *Arrange both dividend and divisor in descending powers of some common letter (called the letter of arrangement).*

Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.

Multiply the divisor by this first term of the quotient and subtract the product from the dividend.

Divide the first term of this remainder by the first term of the divisor for the second term of the quotient, and proceed as before until the remainder vanishes or is of lower degree in the letter of arrangement than the divisor.

When the last remainder is not zero we may express the result of division thus:

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

When the last remainder is zero the dividend is exactly divisible by the divisor. This fact may be expressed as follows:

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{quotient.}$$

The coefficients in the quotient will be rational numbers if those in both dividend and divisor are rational.

EXERCISES

Divide and check the following:

1. $6x^4 + 19x^3y + 11x^2y^2 - 12xy^3 + 6y^4$ by $2x + 3y$.

Solution.

$$\begin{array}{r|rrrrr|l}
2x+3y & 6x^4 & +19x^3y & +11x^2y^2 & -12xy^3 & +6y^4 & 3x^3+5x^2y-2xy^2-3y^3 \\
 & 6x^4 & +9x^3y & & & & \\
\cline{2-3}
 & & 10x^3y & +11x^2y^2 & & & \\
 & & 10x^3y & +15x^2y^2 & & & \\
\cline{3-4}
 & & & -4x^2y^2 & -12xy^3 & & \\
 & & & -4x^2y^2 & -6xy^3 & & \\
\cline{4-5}
 & & & & -6xy^3 & +6y^4 & \\
 & & & & -6xy^3 & -9y^4 & \\
\cline{5-6}
 & & & & & +15y^4 &
\end{array}$$

Result. $3x^3 + 5x^2y - 2xy^2 - 3y^3 + \dfrac{15y^4}{2x+3y}$. ✓

Check. Let $a = b = 1$. Dividend $= 30$, divisor $= 5$, quotient $= 6$; $30 \div 5 = 6$.

2. $6a^2 - 5ab + 3ab^2$ by $3a$.
3. $12x^2 - 5xy + 2y^2$ by $-2y$.
4. $-125a^2b^6c^5$ by $13ab^2c^4$.
5. $13x^2y^{10}z^{15}$ by $-6xy^9z^{10}$.
6. $m^2 - 2mn + 2$ by $m - n$.
7. $3a^2b + 2b - 5b^2$ by $-b + 2$.
8. $x^5 - y^5$ by $x^3 - y^3$.
9. $x^{10} - y^{10}$ by $x^4 - y^4$.
10. $a^4 - b^4$ by $a^2 - 2ab + b^2$.
11. $m^2n - 3(m^2 - 2)n^2 + 3m - 2$ by $2m^2$.
12. $\frac{1}{3}x^2 + \frac{2}{3}xy - \frac{1}{2}xy^2 + 2y^3$ by $\frac{9}{10}x - \frac{5}{4}y$.

13. $2x^ny - 5x^{n-3}y^m + 3y^n$ by x^2.

14. $3m^x - 15mn^{x-1} + 3n^{2x+3}$ by $m^{x-2} + 1$.

15. $a^2 + 5bc + 2b^2 - 3ac + 2ab - c^2$ by $3a - b + 2c$.

16. $21x^ay^b + 12x^{a+2}y^{3b-1} + 3x^b - y^{a+2b}$ by $4x^ay^b$.

17. $x^3 + (a + b + 2c)x^2 + (ab + 2bc + 2ac)x + 2abc$ by $x + a$.

18. $a^5 - a^4b + 3a^3b^2 - 4a^2b^3 - 13ab^4 + 2b^5$ by $a - 2b$.

19. $x^3 + 2x^2y - x^2 + xy^2 - 2xy - y + y^2 - 2$ by $x + y - 1$.

20. $4x^5 - 9x^4y + 2x^3y^2 - 18y^5 + 2xy^4 - 5x^2y^3$ by $x + 10y$.

CHAPTER II

FACTORING

21. Statement of the problem. The operation of *division* consists in finding the quotient when the dividend and divisor are given. The product of the quotient and the divisor is the dividend, and the quotient and the divisor are the factors of the dividend. Thus the process of division consists in finding a second factor of a given expression when one factor is given.

The process of **factoring** consists in finding all the factors of an expression when no one of them is given. This operation is in essence the reverse of the operation of multiplication. We shall be concerned only with those factors that have rational coefficients.

22. Prime factors. An integral expression is **prime** when it contains no integral factors except itself and 1.

It must be remembered that to factor an integral expression means to resolve it into its prime factors.

The methods of this chapter enable one to factor certain algebraic expressions in one letter which are not prime, as well as some of the simpler expressions in two letters.

No attempt is made even to define what is meant by prime factors of expressions which are not rational and integral.

There is no simple operation the performance of which gives assurance that the prime factors of a given expression have been found. Only insight and experience enable one to find prime factors with certainty.

A partial check that may be applied to all the exercises in factoring consists in actually multiplying together the factors that have been found. If the result is the original expression, it is certain that factors have been found, although they may not be prime factors.

23. Monomial factors. By the distributive law, section 10,

$$ab + ac = a(b + c).$$

This affords immediately the type form

$$ab + ac - ad = a(b + c - d).$$

This process is stated in the following

Rule. *Write the largest monomial factor occurring in every term outside a parenthesis which includes the algebraic sum of the remaining factors of the various terms.*

ORAL EXERCISES

Factor:

1. $ab - ac.$
2. $3\,xy + 2\,yz.$
3. $2\,ab^2 + b^2.$
4. $5\,mn + 2\,n^2.$
5. $8\,p^2r - 3\,pr^2.$
6. $2\,ab^2 + 12\,a^2b.$
7. $3\,m - 5\,m^2 + 2\,mn.$
8. $2\,x^2 - 3\,xy + 4\,xy^2.$
9. $4\,gh + 2\,h - 3\,h^2.$
10. $2\,ab - 4\,a^2b.$
11. $15\,m + 5\,m^2.$
12. $2\,xy + 5\,y^2 - 10\,y^3.$
13. $8\,x^2z^4 + 4\,xz.$
14. $5\,y^2 - 25\,xy^3.$
15. $21\,x^2yz^3 - 15\,x^3y^2z.$
16. $a^2b + ab^2 + b^3.$

EXERCISES

Factor:

1. $12\,x^2y - 15\,xy + 21\,xy^2.$

Solution. $12\,x^2y - 15\,xy + 21\,xy^2 = 3\,xy(4\,x - 5 + 7\,y).$

2. $3\,x^2yz^5 + 21\,xy^4z^3 - 13\,x^2yz^2.$
3. $3\,a^5b^7c^{13}d^4 - 12\,a^4b^5c^2d^8.$
4. $36\,a^3 + 24\,a^2b - 12\,a^4b^2.$
5. $2\,m^2np^3 - 12\,mn^2p^5 + 21\,m^4n^2p - 25\,m^2n.$
6. $25\,a^2x^5y^9z^3 - 5\,a^4x^2y^{10}z^2 + 10\,a^2x^9y^8z^4.$

7. $18\,a^2b^5 - 21\,a^5b^2 + 15\,a^{10}b^4 - 3\,a^9b^6$.

8. $21\,m^2n^3q^5 + 63\,m^5n^4q^2 + 7\,m^2n^2q^2 + 14\,m^4n^5q^6$.

9. $2\,a^2bc - 5\,ab^2c^3 - 23\,a^2b^4c^5 + 9\,a^5b^2c$.

10. $18\,x^2y^5z^{10} + 36\,x^4y^8z^6 - 72\,x^2y^5z^8 + 9\,x^2y^2z^4$.

24. Factoring by grouping terms. If in the expression for the distributive law, $ac + bc = (a + b)c$, we replace c by $x + y$, we have

$$a(x + y) + b(x + y) = ax + ay + bx + by.$$

We may then factor the right-hand member as follows:

$$ax + ay + bx + by = a(x + y) + b(x + y) = (a + b)(x + y).$$

This affords the

Rule. *Factor out any monomial expression that is common to each term of the polynomial.*

Arrange the terms of the polynomial to be factored in groups of two or more terms each, such that in each group a monomial factor may be taken outside a parenthesis which in each case contains the same expression.

Write the algebraic sum of the monomial factors that occur outside the various parentheses for one factor, and the expression inside the parentheses for the other factor.

EXERCISES

Factor:

1. $3\,xy - 3\,x^2 + 6\,ay - 6\,ax$.

Solution.
$$\begin{aligned} 3\,xy - 3\,x^2 + 6\,ay - 6\,ax &= 3(xy - x^2 + 2\,ay - 2\,ax) \\ &= 3[x(y - x) + 2\,a(y - x)] \\ &= 3(x + 2\,a)(y - x). \end{aligned}$$

2. $24\,a^2x - ay(32\,a - 18\,x + 24\,y)$.

Solution.
$$\begin{aligned} &24\,a^2x - ay(32\,a - 18\,x + 24\,y) \\ &= 24\,a^2x - 32\,a^2y + 18\,axy - 24\,ay^2 \\ &= 2\,a(12\,ax - 16\,ay + 9\,xy - 12\,y^2) \\ &= 2\,a[4\,a(3\,x - 4\,y) + 3\,y(3\,x - 4\,y)] \\ &= 2\,a(4\,a + 3\,y)(3\,x - 4\,y). \end{aligned}$$

3. $xy + 3y + x + 3$.
4. $3am + 3bm - 3an - 3bn$.
5. $2p^2 + 3pr + 4p + 6r$.
6. $8x^2 + 12xy - 4x - 6y$.
7. $4a^2 + 2ac + 4ab + 2bc$.
8. $ax + bx + cx - ay - by - cy$.
9. $a^px^2 - b^qy^3 - b^qx^2 + a^py^3$.
10. $2x(x - b) - 6ax + 6ab$.
11. $x^2 + 5x - 3ax - 15a$.
12. $5m^2x + 10abnx + 3m^2a + 6a^2bn$.
13. $5x^3 + 2x^2 + 5x + 2$.
14. $m^3a + 2m + 6a + 3m^2a^2$.
15. $24x^7 + 12x^5y^3 - 30y^5 - 60x^2y^2$.
16. $-2m + 10n - 35nx + 7mx$.
17. $45a^3b^2 + 50b^3c^2 + 75a^2bc + 30ab^4c$.
18. $4x^2y^3z^2 - 10x^3y^4z - 5x^4y^8z^2 + 2x^3y^7z^3$.
19. $8m^2n^5p^4 + 12mn^{10}p^4 + 6m^2p^5 + 9mn^5p^5$.
20. $4a^3b + 6ab^2 - 8a^4 - 15b^3 - 3b^3 + 20a^3b$.
21. $15x^3y + 21x^3 - 6x^2y - 10xy^3 - 14xy^2 + 4y^3$.
22. $6ax^2y + 9bxy^2 + 9cxy - 10a^2bx - 15ab^2y - 15abc$.
23. $2ax^3y + 6ax^2 + 4ax^2y + 3x^2y^3 + 9xy^2 + 6xy^3$.
24. $4x^3y^2z^2 + 6x^4y^2z + 10x^4y^2z^6 - 2x^2y^3z^4 - 3x^3y^3z^3 - 5x^3y^3z^8$.
25. $9a^2b^2cxy^2 + 6a^3bc^5xy^2 + 15a^2bcxy^2 + 6ab^3cy + 4a^2b^2c^5y + 10ab^2cy$.

25. Factoring the difference of squares. In section 18 the expression $(a - b)(a + b) = a^2 - b^2$ for the product of the sum and the difference of two terms gives the following type form:

$$a^2 - b^2,$$

and suggests the following

Rule. Extract the square root of each term.

The sum of these square roots is one factor, and their difference is the other.

EXAMPLE

Factor $9a^2x^6y^4 - 16b^8c^2$.

Solution. $9a^2x^6y^4 - 16b^8c^2 = (3ax^3y^2 + 4b^4c)(3ax^3y^2 - 4b^4c)$.

ORAL EXERCISES

Factor:

1. a^2b^2.
2. $x^2 - 9\,b^2$.
3. $25\,x^4 - 16\,y^2$.
4. $36\,n^6 - 81\,n^4$.
5. $p^2 - 64\,r^2$.
6. $x^2 - z^6$.
7. $25 - 16$.
8. $4\,x^2y^4 - 9\,z^2$.
9. $x^4 - 16$.

HINT. Find three factors.

10. $a^8 - b^4$.
11. $x^{2a} - y^{6b}$.
12. $4\,a^{2n} - b^{2m}$.
13. $m^{2n} - 1$.
14. $4\,x^6 - y^2$.

26. Trinomials which are perfect squares. The type form is

$$a^2 \pm 2\,ab + b^2.$$

Factoring, $a^2 \pm 2\,ab + b^2 = (a \pm b)^2$.

ORAL EXERCISES

Separate into binomial factors:

1. $a^2 + 2\,ax + x^2$.
2. $x^2 - 2\,xy + y^2$.
3. $m^2 + 2\,m + 1$.
4. $4\,r^2 - 4\,r + 1$.
5. $x^2 + 4\,xy + 4\,y^2$.
6. $a^2 - 4\,a + 4$.
7. $9\,x^2 + 6\,x + 1$.
8. $a^{2n} + 2\,a^n + 1$.
9. $100\,a^2 - 20\,ab + b^2$.
10. $25\,x^2 - 10\,x + 1$.
11. $4 + 4\,ab + a^2b^2$.
12. $1 - 2\,xy^2 + x^2y^4$.
13. $x^{2a} + 2\,x^ay^b + y^{2b}$.
14. $25\,a^{2x} + 10\,a^xb^y + b^{2y}$.
15. $(a + b)^2 + 2(a + b) + 1$.
16. $(x + y)^2 - 2(x + y)(x - y) + (x - y)^2$.

27. Reduction to the difference of squares. The method of section 26 may be used when the expression to be factored becomes a perfect square by the addition of the square of some expression.

EXERCISES

Factor the following by reduction to difference of squares:

1. $4\,x^4 + y^4$.

Solution.

$$\begin{aligned} 4\,x^4 + y^4 &= 4\,x^4 + 4\,x^2y^2 + y^4 - 4\,x^2y^2 \\ &= (2\,x^2 + y^2)^2 - (2\,xy)^2 \\ &= [(2\,x^2 + y^2) - 2\,xy]\,[(2\,x^2 + y^2) + 2\,xy]. \end{aligned}$$

2. $a^4 + a^2b^2 + b^4$.
3. $x^2 + 4xy + 3y^2$.
4. $4x^8 + y^4$.
5. $x^4 - 13x^2 + 36$.
6. $a^4 + 4$.
7. $m^4 + 81 + 9m^2$.
8. $4x^{4a} + y^{4b}$.
9. $4m^4 + 1$.
10. $x^4 + 5x^2 + 9$.
11. $25a^4 + 9 - 19a^2$.
12. $4m^8 + 3m^4n^4 + 9n^8$.
13. $x^4 + 4x^2 + 16$.
14. $64x^4y^2 + y^2$.

28. The quadratic trinomial. The type form is

$$x^2 + bx + c.$$

Since $$(x + h)(x + k) = x^2 + (h + k)x + hk,$$

it follows that $x^2 + bx + c$ can be factored into the binomial factors $(x + h)(x + k)$ if two numbers h and k can be found which have the sum b and the product c.

ORAL EXERCISES

Factor:

1. $x^2 + 3x + 2$.

Solution. Since the numbers 1 and 2 have 2 for their product and 3 for their sum, the factors of $x^2 + 3x + 2$ are $x + 1$ and $x + 2$.

2. $x^2 + 2x + 1$.
3. $x^2 - 5x - 6$.
4. $a^2 - a - 6$.
5. $m^2 + 2m - 8$.
6. $x^2 + 2x - 3$.
7. $r^2 - 5r + 6$.
8. $x^2 + x - 2$.
9. $x^2 - 8x + 12$.
10. $m^2 - 3m - 10$.
11. $x^2 + 6x + 8$.
12. $x^2 - x - 30$.
13. $a^2 + 2a - 24$.

29. Factors of a quadratic trinomial. In this case we cannot factor by grouping terms immediately, as that method is inapplicable to a polynomial of less than four terms. We observe, however, that in the product of two binomial expressions,

$$(mx + n)(px + q) = mpx^2 + (mq + np)x + nq,$$

the coefficient of x is the sum of two expressions mq and np, whose product is equal to the product of the coefficient of x^2 and the last term; that is,

$$mq \cdot np = mp \cdot nq.$$

Thus, to factor the right-hand member of this equation, we may remove the parenthesis from the term in x and use the principle of grouping terms. Thus

$$\begin{aligned} mpx^2 + (mq + np)x + nq &= mpx^2 + mqx + npx + nq \\ &= mx(px + q) + n(px + q) \\ &= (mx + n)(px + q). \end{aligned}$$

This affords the following

Rule. *Write the trinomial in order of descending powers of x (or the letter in which the expression is quadratic).*

Multiply the coefficient of x^2 by the term not involving x, and find two factors of this product whose algebraic sum is the coefficient of x.

Replace the coefficient of x by this sum and factor by grouping terms.

Factoring a perfect square is evidently a special case under this method. Thus, factor $x^2 + 6x + 9$.

$$1 \cdot 9 = 9. \qquad\qquad 3 + 3 = 6.$$

$$\begin{aligned} x^2 + (3 + 3)x + 9 &= x^2 + 3x + 3x + 9 \\ &= x(x + 3) + 3(x + 3) \\ &= (x + 3)(x + 3) \\ &= (x + 3)^2. \end{aligned}$$

One will usually recognize when a trinomial is a perfect square, in which case the factors may be written down by inspection.

EXERCISES

Factor the following:

1. $2x^2 + 7x + 3$.

Solution. $2 \cdot 3 = 6. \qquad\qquad 6 + 1 = 7.$

$$\begin{aligned} 2x^2 + 7x + 3 &= 2x^2 + (6 + 1)x + 3 \\ &= 2x^2 + 6x + x + 3 \\ &= 2x(x + 3) + (x + 3) \\ &= (2x + 1)(x + 3). \end{aligned}$$

2. $10\,x^2 + 14\,ax - 12\,a^2$.

Solution. $10(-12\,a^2) = -120\,a^2$. $\qquad 20\,a - 6\,a = 14\,a$.

$$\begin{aligned} 10\,x^2 + 14\,ax - 12\,a^2 &= 10\,x^2 + (20\,a - 6\,a)x - 12\,a^2 \\ &= 10\,x^2 + 20\,ax - 6\,ax - 12\,a^2 \\ &= 10\,x(x + 2\,a) - 6\,a(x + 2\,a) \\ &= 2(5\,x - 3\,a)(x + 2\,a). \end{aligned}$$

3. $8\,r^4x^2 - 2\,abr^2x - 21\,a^2b^2$.

Solution. $8\,r^4(-21\,a^2b^2) = -168\,a^2b^2r^4$.

$-14\,abr^2 + 12\,abr^2 = -2\,abr^2$.

$$\begin{aligned} 8\,r^4x^2 - 2\,abr^2x - 21\,a^2b^2 &= 8\,r^4x^2 - (14\,abr^2 - 12\,abr^2)x - 21\,a^2b^2 \\ &= 8\,r^4x^2 - 14\,abr^2x + 12\,abr^2x - 21\,a^2b^2 \\ &= 2\,r^2x(4\,r^2x - 7\,ab) + 3\,ab(4\,r^2x - 7\,ab) \\ &= (2\,r^2x + 3\,ab)(4\,r^2x - 7\,ab). \end{aligned}$$

4. $x^2 + x - 6$.
5. $x^2 + 3\,ax - 10\,a^2$.
6. $6\,a^2 - 17\,ab + 5\,b^2$.
7. $6\,x^2 + 19\,xy + 15\,y^2$.
8. $15\,a^2 + 19\,abx - 56\,b^2x^2$.
9. $195\,m^2 + 17\,mp - 154\,p^2$.
10. $36\,x^2 - 5\,xy - 50\,y^2$.
11. $3\,x^2 - 12\,xy - 15\,y^2$.
12. $42\,x^2 + 25\,xy - 23\,y^2$.
13. $84\,a^2 - 271\,ab + 195\,b^2$.
14. $x^2 + xy - 12\,y^2$.
15. $10\,m^2 - 7\,mn - 12\,n^2$.
16. $3\,x^2 + 2\,xy - y^2$.
17. $18\,x^2 - 3\,xy - 10\,y^2$.
18. $14\,a^2 - 18\,ab + 4\,b^2$.
19. $50\,x^2 + 35\,xy + 6\,y^2$.
20. $24\,m^2 - 34\,mn - 65\,n^2$.
21. $14\,x^4 + 33\,x^2y + 18\,y^2$.
22. $10\,x^4 + bx^2y - 21\,b^2y^2$.
23. $24\,a^4 - 26\,a^2b - 15\,b^2$.
24. $8\,x^{2a} + 22\,x^ay + 15\,y^2$.
25. $15\,x^{2a} + 44\,x^ay^b - 3\,y^{2b}$.
26. $12\,a^2 + 25\,ab + 12\,b^2$.
27. $6\,a^2x^4 - 5\,abx^2y - 50\,b^2y^2$.
28. $50\,x^{2m} + 13\,x^my^n - 6\,y^{2n}$.
29. $30\,a^2 + 259\,ab + 75\,b^2$.
30. $15\,a^4b^2c^2 + 23\,a^2bcde^2 - 90\,d^2e^4$.

30. Factoring binomials of the form $a^n \pm b^n$. By section 20 we can show that

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}). \qquad (1)$$

$$a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + b^{n-1}). \qquad (2)$$

The first form is valid for any integral values of n, whether even or odd. Equation (2) is valid only for odd values of n. The binomial $a^n + b^n$ is prime when n is even.

One can factor by inspection any binomial of the given form by reference to these equations.

EXERCISES

Factor the following:

1. $x^{10} - y^{10}$.

Solution. $x^{10} - y^{10} = (x^5 - y^5)(x^5 + y^5)$

$$= (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)(x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4).$$

2. $x^4 - 16$.
3. $x^5 - 1$.
4. $x^5 + 1$.
5. $x^9 + y^9$.
6. $a^{13} - b^{13}$.
7. $a^{15} + b^{15}$.
8. $a^2b^3 + a^5$.
9. $- x^2 + a^4b^6$.
10. $25\,m^{12} - 9\,m^6$.
11. $ay^6 - 125\,a$.
12. $4\,x^5 - 500\,x^2y^3$.
13. $27\,x^6y^9 + 64\,y^3$.
14. $8\,a^3b^5 - b^2$.
15. $64\,x^8y^2 - a^6b^4$.
16. $1600\,x^2y^4 - 25\,x^4y^6$.

31. Highest common factor. If two polynomials have the same expression as a factor, this expression is said to be their **common factor.**

The product of the common prime factors of two polynomials is called their **highest common factor.**

The same common prime factor may occur more than once. Thus $(x-1)^2(x+1)$ and $(x-1)^2(x-2)^2$ have $(x-1)^2$ as their highest common factor.

32. Highest common factor of two polynomials. The process of finding the highest common factor is performed according to the following

Rule. Factor the polynomials. The product of the common prime factors is their highest common factor.

EXERCISES

Find the highest common factor of the following:

1. $4a^3x - 2a^2bx - 2ab^2x$ and $2a^3 + a^2b - 2ab^2 - b^3$.

Solution.
$$\begin{aligned} 4a^3x - 2a^2bx - 2ab^2x &= 2ax(2a^2 - ab - b^2) \\ &= 2ax(a-b)(2a+b). \\ 2a^3 + a^2b - 2ab^2 - b^3 &= a^2(2a+b) - b^2(2a+b) \\ &= (a+b)(a-b)(2a+b). \end{aligned}$$

The highest common factor is, then, $(a-b)(2a+b)$.

2. $2a^3 - 2ax^2$ and $2a^3 + 4a^2x + 2ax^2$.

3. $3x^2 + 2xy - y^2$ and $4x^2 + 7xy + 3y^2$.

4. $2a^2x^2 + 2a^2xy - xy - y^2$ and $x^3 - xy^2 + x^2y - y^3$.

5. $8x^2 + 2xy - 3y^2$ and $20x^2 + 23xy + 6y^2$.

6. $8a^3 - 4a^2b - 4ab^2$ and $6a^2 + 11ab + 4b^2$.

7. $2m^3n + 6mn + 2m^2n^2 + 6n^2$ and $6m^3 + 18m - 3n - m^2n$.

8. $6ab + 3ay + xy + 2bx$ and $6a^2 + 11ax + 3x^2$.

9. $a^2x^3 + abx^2 + acx + ax^2 + bx + c$ and $a^2x^4 + 2acx^2 - b^2x^2 + c^2$.

10. $16ax^2 + 8bx^2 - 12axy - 6bxy - 10ay^2 - 5by^2$ and $12x^2 + 8x - 15xy - 10y$.

EXERCISES

Factor:

1. $2y^2 + xy - x^2$.

2. $6a^2b^2c + 15ab^3c - 30ab^2c^2 + 9ab^2c$.

3. $156x^2 + 113xy - 119y^2$.

4. $15m^2 - 14mn - 16n^2$.

5. $10x^2 + 13x - 3$.

6. $8x^2 + 10x + 3$.

7. $x^7 + y^7$.

8. $2ax - bx + cx + 4ay - 2by + 2cy - 2az + bz - cz$.

9. $12a^4b^2 - 37a^2bmn - 30m^2n^2$.

10. $3a^2 + 7ab - 6b^2$.

11. $6x^3 - 10xy^3 - 15y^4 + 9x^2y$.

12. $9a^4 + 8a^2b^2 + 4b^4$.

13. $30a^2b + 14a^2b^2 + 45ab^2 + 21ab^3$.

14. $x^3 + 3x^2 + 3x + 1 - 8y^3$.
15. $m^3 + 6m^2n + 5mn^2$.
16. $6mx + 10my + 14mc - 3ax - 5ay - 7ac - 24xz - 40yz - 56cz$.
17. $150x^4y^8z^5 + 26x^3y^4z^5 - 750x^4y^6z^5 - 130x^3y^2z^5$.
18. $8x^3yz^3 - 10x^2y^2z^5 + 6x^4yz^4 + 10x^2yz^3$.
19. $10x^4 - 5x^3 - 5x^2$.
20. $50x^2 - 27xy - 153y^2$.
21. $26x^2 - 129xy - 77y^2$.
22. $2x^2 + 5xy + 10xz - 6x - 15y - 30z$.
23. $30x^3y^4z^6 - 225x^4yz^5 + 75x^2y^3z^6 - 45x^2yz^5$.
24. $x^3 + y^3$.
25. $26a^3b + 65a^2b - 130a^2b^2 + 65a^4b$.
26. $x^2 - y^2 + xz + yz$.
27. $20a^2x^2 - 44abx - 15b^2$.
28. $15a^2 + 13a + 2$.
29. $15x^2 - 91xy + 132y^2$.
30. $63x^2 + 222xy + 40y^2$.
31. $216m^3 - n^6$.
32. $a^{12} - b^{12}$.
33. $6a^3b^3c + 10a^2b^4c^2 - 6a^2b^3c^2 + 2a^2b^3c$.
34. $2a^2 + 5ab + 2b^2$.
35. $a^{10} - b^{10}$.
36. $8a^3 + 27b^6$.
37. $156x^2 + 125xy + 25y^2$.
38. $25x^4 + x^2 + 1$.
39. $2a^2 - ab^2 - 3b^4$.
40. $x^{15} + y^{15}$.
41. $x^{25} + y^{25}$.
42. $75a^2 + 50a + 4b + 2c + 3ac + 6ab$.
43. $6a^3b - 27a^2b^2 - 15ab^3$.
44. $25a^4 - 60a^2b + 36b^2$.
45. $24a^3 + 66a^2b + 45ab^2 - 75b^2 - 40a^2 - 110ab$.
46. $18x^2 + 450xy - 5000y^2$.
47. $4x^4 + 16x^2 + 15$.
48. $10ab^2xy^3 - 4a^3x^3y + 3a^2bx^2y^2$.
49. $8a^2 + 22a + 15$.
50. $15a^2 + 35ab + 10b^2 + 2a + 4b$.
51. $2a^2 - 3ab - 5b^2$.

52. $x^4 + 4x^2 + 16.$

53. $3a^2 - 11ab - 20b^2.$

54. $3x^2 - xy - 10y^2.$

55. $x^5 - y^5.$

56. $x^2 + 2ax + a^2 - 4x^2 - 4ax - a^2.$

57. $25x^2y^6 - 36x^4y^8.$

58. $x^2 + 5x - 14.$

59. $a^4 - b^4.$

60. $x^2 + 6xy + 8y^2.$

61. $4x^2y + 10xy^2 + 8x^5y^3 + 6xy.$

62. $2x^2 - 3xy - 5y^2.$

63. $6x^2 + 15x + 20y + 8xy.$

64. $6x^3 + 17x^2 + 12x - 6x^2y - 17xy - 12y.$

65. $x^{14} - y^{14}.$

66. $3m^2 - mn - 2n^2.$

67. $315x^4 - 171x^2y - 44y^2.$

68. $6ax^4 + 14ax^3y - 12ax^2y^2.$

69. $x^5 + y^5.$

70. $6x^2 + 3xy - 30y^2.$

71. $6ax^2 + 10bx - 3abxy - 5b^2y.$

72. $24ax^2 + 134axy - 100ay^2.$

73. $45x^3 + 75x^2y + 6xy + 10y^2.$

74. $6m^3n^3q + 9mn^3q^2 - 15mn^6q - 3mn^3q.$

75. $50x^3 + 35ax^2 + 100x^2 + 6a^2x + 70ax + 12a^2.$

76. $16abx^2 + 8abx - 24ab.$

Perform the indicated operations:

77. $95 \cdot 105.$

Solution.

$$\begin{aligned} 95 &= 100 - 5. \\ 105 &= 100 + 5. \\ 95 \times 105 &= 10{,}000 - 25 \\ &= 9975. \end{aligned}$$

78. $32 \cdot 28.$

79. $121 \cdot 119.$

80. $37 \cdot 39 + 52 \cdot 74.$

HINT. Divide by a monomial factor.

81. $12 \cdot 3 + 72 \cdot 4 - 21 \cdot 6.$

82. $25 \cdot 25 + 25 \cdot 7 - 25 \cdot 4.$

33. Least common multiple. The least common multiple of two or more polynomials is the polynomial of least degree that contains them as factors. We may find the least common multiple of several polynomials by the following

Rule. *Multiply together all the factors of the various polynomials, giving to each factor the greatest exponent with which it appears in any of the polynomials.*

EXERCISES

Find in factored form the least common multiple of the following:

1. $(x^2 - ax - 2a^2)$, $(x^2 - a^2)$, and $(x^2 + 2ax + a^2)$.

Solution. $x^2 - ax - 2a^2 = (x - 2a)(x + a)$.

$$x^2 - a^2 = (x - a)(x + a).$$

$$x^2 + 2ax + a^2 = (x + a)^2.$$

Therefore the least common multiple $= (x + a)^2(x - 2a)(x - a)$.

2. $12a^2$, $9xy^2$, and $4y^3$.

3. $5ab$, $15b^2$, and $3ax$.

4. $(x - a)$, $(x^2 - 2ax + a^2)$, and $(x^2 - a^2)$.

5. $(x^3 - y^3)$ and $(x^2 - y^2)$.

6. $(2x^2 - 2y^2 + 3xy)$ and $(x^2 + 2xy)$.

7. $(9x^2 + 30xy + 25y^2)$ and $(9x^2 - 25y^2)$.

8. $(x^2 + 3xy + 2y^2)$, $(x^2 + xy - 2y^2)$, and $(x^2 + xz - xy - yz)$.

9. $(x^2 + 2x - 8)$, $(x^2 - 5x + 6)$, and $(x^2 + x - 12)$.

10. $(x^3 - 2ax^2 - a^2x + 2a^3)$ and $(x^3 + ax^2 - 4a^2x - 4a^3)$.

11. $(2x^2 - x - 10)$, $(2x^3 + 3x^2 - 12x - 18)$, and $(6x^2 - 15x - 5y + 2xy)$.

CHAPTER III

FRACTIONS

34. General principles. The symbolic statements of the rules for the addition, subtraction, multiplication, and division of algebraic fractions are the same as the statements of the corresponding operations on numeric fractions given in (2), (3), and (4), section 6. This is immediately evident if we keep in mind the fact that algebraic expressions are symbols for numbers and that if the letters are replaced by numbers, the algebraic fraction becomes a numeric fraction.

35. ***Principle I.*** *Both numerator and denominator of a fraction may be multiplied (or divided) by the same expression without changing the value of the fraction.*

This follows from (5), section 6.

36. ***Principle II.*** *If the signs of both numerator and denominator of a fraction be changed, the sign of the fraction remains unchanged.*

This follows from Principle I, when we multiply both numerator and denominator by -1.

37. ***Principle III.*** *If the sign of either numerator or denominator (but not both) be changed, the sign of the fraction is changed.*

This follows from (6), section 6.

38. Reduction. A fraction is said to be **reduced to its lowest terms** when its numerator and denominator have no common factor. We effect this reduction by following the

Rule. *Divide both numerator and denominator by their highest common factor.*

EXERCISES

Reduce the following fractions to their lowest terms:

1. $\dfrac{48\,ax^2 + 152\,axy + 120\,ay^2}{8\,ax^2 - 12\,axy - 36\,ay^2}$.

Solution.

$$\frac{48\,ax^2 + 152\,axy + 120\,ay^2}{8\,ax^2 - 12\,axy - 36\,ay^2} = \frac{8\,a(3\,x + 5\,y)(2\,x + 3\,y)}{4\,a(x - 3\,y)(2\,x + 3\,y)} = \frac{2(3\,x + 5\,y)}{(x - 3\,y)}.$$

2. $\dfrac{a^2 - b^2}{2\,a + 2\,b}$.

3. $\dfrac{x^2 + 2\,x + 1}{x^2 - 1}$.

4. $\dfrac{x^2 - 5\,x - 24}{x^2 - 16\,x + 64}$.

5. $\dfrac{2\,a^2 + 5\,ay + 6\,ax}{2\,a^2 + 3\,ay + 6\,ax - 5\,y^2 - 6\,xy}$.

6. $\dfrac{6\,x^2 - 7\,xy - 20\,y^2}{4\,x^2 - 25\,y^2}$.

7. $\dfrac{x^3 - 2\,x^2 + 5\,x - 10}{2\,x^4 + 7\,x^2 - 15}$.

8. $\dfrac{6\,a^3 - 4\,ab^2 - 2\,a^2b}{20\,a^3 - 8\,a^3b - 14\,a^2b + 4\,a^2b^2 - 6\,ab^2 + 4\,ab^3}$.

9. $\dfrac{a^2 + b^2 - c^2 + 2\,ab}{a^2 - b^2 + c^2 + 2\,ac}$.

10. $\dfrac{ax + a + 2\,x + 2}{x^2 - 1}$.

11. $\dfrac{2\,a^2b + 2\,ab^2 - 2\,abc}{3\,bc^2 - 3\,b^2c - 3\,abc}$.

12. $\dfrac{10\,x^2 - 23\,xy + 9\,y^2}{15\,x^2 - 17\,xy - 18\,y^2}$.

13. $\dfrac{x^8 + 3\,x^2 + 3\,x + 1}{x^2 + 2\,x + 1}$.

14. $\dfrac{14\,a^2b + 10\,ab^2 + 4\,a^3}{2\,a^3 - 2\,ab^2}$.

15. $\dfrac{26\,x^2 - 3\,xy - 5\,y^2}{16\,x^2 - 26\,xy + 9\,y^2}$.

39. Equivalent fractions. Two fractions are **equivalent** when one can be obtained from the other by multiplying or by dividing both of its terms by the same expression.

Two fractions having unlike denominators cannot be added, nor can one be subtracted from the other, until they have been reduced to respectively equivalent fractions having like denominators.

To change two or more fractions (in their lowest terms) to respectively equivalent fractions which may conveniently be added or subtracted, we have the

Rule. *Find the least common multiple of the various denominators.*

Multiply both numerator and denominator of each fraction by the expression which will make the new denominator the least common multiple of the denominators.

If the least common denominator is not easily obtained, the product of the denominators may be used. The result, however, will not be in its lowest terms unless one uses the least common denominator.

EXERCISES

Express the following as equivalent fractions having the same denominator:

1. $\frac{3}{x-1}$, $\frac{4}{x^2-1}$, and $\frac{3x+2}{2x-2}$.

Solution. The least common multiple of the denominators is

$$2(x-1)(x+1).$$

The fractions then are

$$\frac{6x+3}{2x^2-2},\ \frac{8}{2x^2-2},\ \text{and}\ \frac{3x^2+5x+2}{2x^2-2}.$$

2. $\frac{x^2}{4+y}$, $\frac{2z}{16-y^2}$, and $\frac{3y}{4-y}$.

3. $\frac{3}{a}$, $\frac{2a}{b+3}$, and $\frac{5+b}{2b^2+7b+3}$.

4. $\frac{2x}{5y+z}$, $\frac{3}{5y^2-4yz-z^2}$, and $\frac{4}{y^2-z^2}$.

5. $\frac{2x+3}{4x^2-9}$, $\frac{5}{4x^2-6x}$, and $\frac{3x}{2x^2-2xy}$.

6. $\frac{5b}{2ab-10b^2}$, $\frac{3ab^3}{a^2-25b^2}$, and $\frac{21a}{2ab+10b^2}$.

7. $\frac{3xy}{2x+1}$, $\frac{2}{4x^2+4x+1}$, and $\frac{5x}{4x^2-1}$.

8. $\frac{m}{bm+an}$, $\frac{n}{acm^3-am^2n}$, and $\frac{p}{bcm^2+acmn-bmn-an^2}$.

40. Addition of fractions. This operation we perform according to the following

Rule. *Reduce the fractions to be added to their least common denominator.*

Add the numerators for the numerator of the sum, and take the least common denominator for its denominator.

41. Subtraction of fractions. This operation we perform according to the following

Rule. *Reduce the fractions to their least common denominator.*

Subtract the numerator of the subtrahend from that of the minuend for the numerator of the result, and take the least common denominator for its denominator.

42. Multiplication of fractions. This operation we perform according to the following

Rule. *Multiply the numerators together for the numerator of the product, and the denominators for its denominator.*

43. Division of fractions. This operation we perform according to the following

Rule. *Invert the terms of the divisor and multiply by the dividend.*

REMARK. Since a fraction is a means of indicating division, $\frac{a}{b} \div \frac{c}{d}$ and $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ are two expressions for the same thing.

After performing the operations indicated in any of the four preceding rules, the resulting fraction should be reduced to its lowest terms.

The results may be checked by substituting numeric values for the letters in the expressions.

When unity is divided by a number the resulting expression is said to be the **reciprocal** of the original number.

Thus $1 \div 2 = \frac{1}{2}$, and $\frac{1}{2}$ is the reciprocal of 2.
Similarly $1 \div \frac{2}{3} = \frac{3}{2}$, and $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$.

EXERCISES

Perform the indicated operations, bring the results into their simplest forms, and check as directed by the teacher:

1. $$\frac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}.$$

Solution.

$$\frac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}=\frac{\dfrac{(a+b)^2+(a-b)^2}{a^2-b^2}}{\dfrac{(a+b)^2-(a-b)^2}{a^2-b^2}}$$

$$=\frac{(a+b)^2+(a-b)^2}{a^2-b^2}\cdot\frac{a^2-b^2}{(a+b)^2-(a-b)^2}$$

$$=\frac{2\,a^2+2\,b^2}{4\,ab}=\frac{a^2+b^2}{2\,ab}.$$

Check. Let $a=2$ and $b=3$.

Then
$$\frac{\dfrac{2+3}{2-3}+\dfrac{2-3}{2+3}}{\dfrac{2+3}{2-3}-\dfrac{2-3}{2+3}}=\frac{13}{12},$$

and
$$\frac{4+9}{2\cdot 2\cdot 3}=\frac{13}{12}.$$

2. $\dfrac{\frac{12}{5}-1}{\frac{4}{5}+2}.$

3. $\dfrac{x}{ab}-\dfrac{y}{ac}-\dfrac{z}{bc}.$

4. $\dfrac{1}{m+n}+\dfrac{1}{m-n}.$

5. $\dfrac{4\,x-3}{x+1}+\dfrac{2\,x+5}{x-3}.$

6. $\dfrac{x+1}{1-x}+\dfrac{3}{x+2}.$

7. $\dfrac{2\,x+1}{4\,x^2-4\,x+1}-\dfrac{2\,x-1}{4\,x^2+4\,x+1}.$

8. $\dfrac{3}{2\,x+4}+\dfrac{5}{10\,x-2}.$

9. $\dfrac{2\,x}{5\,x+15}-\dfrac{5\,x}{10\,x+30}.$

10. $\dfrac{m-n}{m+n}\cdot\dfrac{2\,m+5\,n}{m^2-n^2}.$

11. $\dfrac{2\,x}{5\,x+1}-\dfrac{21}{13}.$

12. $\dfrac{mn-p}{4\,m^2-4\,mn}\div\dfrac{mp+n}{5\,mn-5\,n^2}.$

13. $\dfrac{2}{16\,x^2+8\,x+1}+\dfrac{13}{4\,x^2+13\,x+3}.$

14. $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}+\dfrac{2x}{x^2-y^2}.$

15. $\dfrac{x^2+x(a+b)+ab}{x^2-x(a+b)+ab}\cdot\dfrac{x^2-a^2}{x^2-b^2}.$

16. $\dfrac{4}{6a^2-7a-20}+\dfrac{5}{6a^2+7a-20}.$

17. $\dfrac{5z}{6a}+\dfrac{13z}{14a}+\dfrac{18z}{35a}-\dfrac{2z}{15a}.$

18. $\dfrac{4x}{x^2-x-6}\div\dfrac{2x}{x^2-5x+6}.$

19. $\dfrac{a(a-x)}{a^2+2ax+x^2}\cdot\dfrac{a(a+x)}{a^2-2ax+x^2}.$

20. $\dfrac{m-2n}{m+2n}\div\dfrac{m^2-4mn+4n^2}{m^2+4mn+4n^2}.$

21. $\dfrac{4}{2x+y}+\dfrac{5}{3x-y}+\dfrac{3}{6x^2+xy-y^2}.$

22. $\dfrac{xy}{x^2y+4xy}\cdot\dfrac{x^2-16}{2xy^2+3xy}.$

23. $\dfrac{2x-1}{4x^2+2x}-\dfrac{2x+1}{6x^2-3x}.$

24. $\dfrac{x^2-\dfrac{1}{x}}{x+\dfrac{1}{x}+1}.$

25. $\dfrac{x^3-1}{x^2-1}+\dfrac{4}{x+1}.$

26. $\dfrac{\left(\dfrac{x}{y}\right)^2-1}{\left(\dfrac{x}{y}\right)^3-1}.$

27. $\dfrac{\dfrac{1}{x+1}+\dfrac{1}{1-x}}{\dfrac{1}{-x+1}-\dfrac{1}{1+x}}.$

28. $\dfrac{2x+5y}{10x^2-13xy-3y^2}\div\dfrac{2x-5y}{10x^2+37xy+30y^2}.$

29. $\dfrac{m+2}{m^2-4m+4}\cdot\dfrac{m-2}{m^2+4m+4}.$

30. $\left(\dfrac{a^2+c^2}{a^2-c^2}-\dfrac{a^2-c^2}{a^2+c^2}\right)\div\left(\dfrac{a+c}{a-c}-\dfrac{a-c}{a+c}\right).$

31. $\dfrac{2x+5y+3}{8}-\dfrac{4x-5y+2}{3}+\dfrac{x+2}{6}.$

32. $\dfrac{4m+2n+1}{3m}+\dfrac{2m^2+2mn+4m}{3n}.$

33. $\dfrac{3\,xyz}{yz + xz + xy} - \dfrac{\dfrac{x-1}{x} + \dfrac{y-1}{y} + \dfrac{z-1}{z}}{\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}}.$

34. $\dfrac{1}{6\,a} + \dfrac{a - 2\,b}{3\,ab} - \dfrac{3}{4\,b} - \dfrac{3\,a - 4\,c^2}{8\,ac^2} + \dfrac{9}{8\,c^2} + \dfrac{5\,c^2 - 5\,b^2}{12\,bc^2}.$

35. $\dfrac{1}{x-1} - \dfrac{1}{x+1} + \dfrac{1}{(x-1)^2} + \dfrac{2}{(x+1)^2} - \dfrac{4}{x^2-1} - \dfrac{4}{(x^2-1)^2}.$

Write the following expressions in two other ways:

36. $-\dfrac{x}{y}.$

37. $\dfrac{2\,a}{5\,c - 1}.$

38. $-\dfrac{-z^2}{-3\,yx}.$

39. $\dfrac{3\,x + 5\,y}{-9\,x^2 - 25\,y^2 + 30\,xy}.$

Show that

40. $2\,x + 5\,b + 10\,a = 3\,b - 3\,x + 3(2\,x + 3\,a)$ when $x = a + 2\,b$.

41. $\dfrac{e^{3x} - e^{-3x}}{e^x - e^{-x}} = 5\tfrac{1}{4}$ when $e^{2x} = 4$.

42. $\dfrac{a}{b} - \dfrac{c}{d} = \dfrac{16}{x^4 + 4\,x^2 + 16}$ when $a = x + 2$, $b = x^2 + 2\,x + 4$, $c = x - 2$, and $d = x^2 - 2\,x + 4$.

Reduce the following to their least common denominator:

43. $\frac{2}{5}$, $-\frac{4}{3}$, and $\frac{1}{2}$.

44. $\frac{11}{13}$, $\frac{12}{7}$, and $\frac{5}{19}$.

45. $\dfrac{13}{16}$, $\dfrac{3}{10}$, and $\dfrac{2}{3\,a}$.

46. $\dfrac{5}{ac}$, $\dfrac{2}{ab^2}$, and $\dfrac{4}{a^2c}$.

47. $\dfrac{2\,x + y}{3\,x}$, $\dfrac{5\,x - 2\,y}{xy}$, and $\dfrac{x + y}{4}$.

48. $\dfrac{4}{x^2 + 4\,x}$ and $\dfrac{5}{x^2 - 16}$.

49. $\dfrac{2\,x}{3\,x + 2\,y}$, $\dfrac{3\,y}{9\,x^2 + 12\,xy + 4\,y^2}$, and $\dfrac{4}{9\,x^2 - 4\,y^2}$.

50. $\dfrac{x - 1}{x^2 - 7\,x + 6}$, $\dfrac{2\,x}{5\,x^2 - 30\,x}$, and $\dfrac{4}{5\,x^2 - 5\,x}$.

CHAPTER IV

RATIO, PROPORTION, AND VARIATION

44. Ratio. The **ratio** of one of two numbers to the other is the result of dividing the first number by the second.

The ratio of a to b is often denoted by $a : b$ or $\frac{a}{b}$. The dividend or numerator in this implied division is called the *antecedent*, the divisor or denominator is called the *consequent*.

Since a ratio is merely a fraction written in a slightly different form, one may always write it as a fraction, and apply the usual operations with fractions as given in Chapter III.

45. Proportion. A **proportion** is an expressed equation between two equal ratios.

Thus $a : b = c : d$ may be written $\frac{a}{b} = \frac{c}{d}$, and treated like any other equation. The terms a and d are often referred to as the *extremes* of a proportion, while b and c are often called the *means*.

Ratios and proportions should always be expressed in fractional and equational form and treated accordingly.

46. Theorem on proportions. The following theorem, which finds considerable application in algebra, is usually expressed by means of the terms defined above.

Theorem. If a number of ratios are equal, the sum of any number of antecedents is to any one antecedent as the sum of the corresponding consequents is to the corresponding single consequent.

Let $$a : b = c : d = e : f = g : h,$$

or $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}.$$

To prove $$\frac{a+c+e}{g} = \frac{b+d+f}{h}.$$

If $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=r,$$

then $$a=br,$$
$$c=dr,$$
$$e=fr,$$
and $$g=hr.$$

Dividing the sum of the first three equations by the last,

$$\frac{a+c+e}{g}=\frac{b+d+f}{h}.$$

47. Mean proportion. The **mean proportional** between two numbers a and c is the number b such that

$$a:b=b:c.$$

That is, $$\frac{a}{b}=\frac{b}{c},$$

or $$b^2=ac,$$

and $$b=\sqrt{ac}.$$

EXERCISES

If $a:b=c:d$, prove the following:

1. $\dfrac{a^2-b^2}{b^2}=\dfrac{c^2-d^2}{d^2}$.

Solution. $$\frac{a}{b}=\frac{c}{d}. \qquad (1)$$

(1) + 1, $$\frac{a}{b}+1=\frac{c}{d}+1,$$

$$\frac{a+b}{b}=\frac{c+d}{d}. \qquad (2)$$

(1) − 1, $$\frac{a}{b}-1=\frac{c}{d}-1,$$

$$\frac{a-b}{b}=\frac{c-d}{d}. \qquad (3)$$

(2) · (3), $$\frac{a^2-b^2}{b^2}=\frac{c^2-d^2}{d^2}.$$

2. $5a:b=5c:d$.

3. $a^2:b^2=c^2:d^2$.

4. $ma:mb=nc:nd$.

5. $b:a=d:c$.

6. $a:c=b:d$.
8. $a-b:a=c-d:c$.
7. $a+b:a=c+d:c$.
9. $a+b:a-b=c+d:c-d$.

10. $a+b:c+d=a:c$.

11. $(a^2-b^2):2ab=(c^2-d^2):2cd$.

12. $\frac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}}=\frac{a}{b}$.

13. $\frac{a^2+ab+b^2}{c^2+cd+d^2}=\frac{a^2-ab+b^2}{c^2-cd+d^2}$.

14. $\frac{5a+b}{5a-b}=\frac{5c+d}{5c-d}$.

15. Find the mean proportional between a^2 and b^2.

16. Find the mean proportional between a^2-b^2 and c^2-d^2.

48. Direct variation. The number x is said to **vary directly** as the number y when the ratio of x to y is constant. This we symbolize by

$$\frac{x}{y}=k, \quad \text{or} \quad x=ky, \tag{1}$$

where k is a constant.

Thus if a man walks at a uniform speed, the distance that he travels varies directly as the time. That is, $d=kt$. In this case the constant k is the speed of the man.

If the length of the altitude of a number of triangles is constant, the areas of the triangles vary as their bases. That is, $A=kb$. In this case k is equal to one half the constant altitude.

The volume of a sphere varies as the cube of the radius. That is, $V=kr^3$. In this case $k=\frac{4\pi}{3}$.

49. Inverse variation. The number x is said to **vary inversely** as the number y when x varies directly as the reciprocal of y. Thus x varies inversely as y when

$$x=\frac{k}{y}, \quad \text{or} \quad xy=k. \tag{2}$$

Thus in triangles of fixed area, the bases b vary inversely as the altitudes a. That is, $ab = k$. In this case k is equal to twice the constant area of the triangles. Expressing this in somewhat less rigorous terms, one might say that if several triangles have the same area, then the longer the base is, the shorter must be the altitude.

If one walks a fixed distance, the time t required varies inversely as the speeds. That is, $st = k$. In this case k is the fixed distance. Again expressing this fact somewhat roughly, one might say that the faster a man walks in covering a certain distance, the less time it takes him.

The intensity of light at a given point varies inversely as the square of the distance from the source of light to the point in question. If L denotes the intensity of the light and d the distance of the light from the point, then $L = \frac{k}{d^2}$, or $Ld^2 = k$.

50. Joint variation. The number x is said to **vary jointly** as y and z when it varies directly as the product of y and z. Thus x varies jointly as y and z when

$$\frac{x}{yz} = k, \quad \text{or} \quad x = kyz. \tag{3}$$

Thus the amount of work accomplished on a job might vary jointly as the number of men employed and the number of days that they work. That is, $W = kMt$, where M denotes the number of men, and t represents the length of time they worked. In this case the meaning of the constant k is not clear until one knows more about the situation. In any case, k stands for the amount of work which one man does in a unit of time, since when $M = t = 1$ we find that $W = k$. A man's wages vary jointly as the number of days that he works and the amount he receives per day.

The number x is said to vary directly as y and inversely as z when x varies directly as $\frac{y}{z}$. Thus the force of the attraction between two bodies in space varies directly as the product of their masses and inversely as the square of their distance apart. If m represents the product of the masses of two bodies, d their distance apart, and G the force of attraction, then

$$G = \frac{km}{d^2}, \quad \text{or} \quad \frac{Gd^2}{m} = k, \quad \text{or} \quad Gd^2 = km.$$

EXERCISES

State the following variations in your own words, but so as to indicate which of the three types of variation is represented in each exercise. In each case k denotes a constant, while the other letters denote variables.

1. $a = kb$.
2. $M = \frac{k}{n}$.
3. $L = kR^2$.
4. $xr = ky$.
5. $V = kl^3$.
6. $S = kt^2$.
7. $C = kr$.
8. $V = kI$.
9. $W = kI^2$.
10. $V = \sqrt{kh}$.
11. $E = kMV^2$.
12. $A = P(1 + r)^2$.

Express the following statements in symbolic form:

13. The volume V of a prism varies jointly as the height and the square of a side of the base.

14. The number a varies directly as the square of b, and inversely as the sum of c and d.

15. The age A of a tree can be told from (varies as) the diameter d and the height h.

51. Applications. The foregoing exercises have to do merely with the statement of the laws or formulas which govern various situations. In each of them the constant k appears. In order to apply the formulas to specific cases it is necessary to determine the value of k. In every instance this can be done if we know the values of the variables for a single particular case. Thus if in the formula $A = kb$ (section 48) we know that when the base of one of the triangles is 6 inches the area is 12 square inches, we are led immediately to the expression $12 = 6\,k$, or $k = 2$. Then, having found the value of k once for all, we are able to obtain the value of A corresponding to any value of b.

PROBLEMS

By use of the method of section 51, solve the following problems:

1. The volume of a sphere varies as the cube of the radius. If the volume of the sphere is 4.19 when the radius is 1, find the volume when the radius is 2; when it is 4.

2. The area of a circle varies as the square of the diameter. If the area is 3.14 when the diameter is 2, what is the area when the diameter is 10?

3. The weight of a body varies inversely as the square of the distance of the body from the center of the earth. If a body weighs 100 pounds when it is 4000 miles from the center of the earth, how much will it weigh when it is 200 miles farther away?

4. The cost of a certain type of building varies jointly as the size and the number of rooms. If a building 50 feet long, 30 feet wide, and 20 feet high contained 8 rooms and cost $20,000, how much would a building 50 feet by 50 feet by 30 feet and containing 12 rooms cost?

5. An automobile is worth an amount inversely proportional to its age. If an automobile was worth $1000 when it was 2 years old, how much will it be worth when it is 6 years old?

6. A body falling from rest falls with a speed which varies directly as the time the body has fallen. If a body has a speed of 32 feet per second at the end of the first second, what will be its speed at the end of the third second?

7. The distance fallen by a body varies directly as the square of the time the body has been falling. If a body falls 64 feet in the first two seconds, how far will it fall in 9 seconds?

8. A body falling from a cliff takes 10 seconds to reach the bottom. How high is the cliff?

9. If I can travel 200 miles in one day at a certain speed, how far can I go in $2\frac{1}{2}$ days at the same speed?

10. If three men work 8 hours a day and do a certain job in 4 days, how long will it take 5 men to do the same job if they work only 6 hours a day?

11. A motion-picture screen is 100 feet from the projector. Where must the screen be placed so that the light will appear twice as bright on the screen?

12. A man reads a book easily when the light is 3 feet from the page. Where must he place a light three times as bright in order to get the same illumination?

13. The weight of men of uniform build varies as the cube of their height. If a man 6 feet tall weighs 170 pounds, what would you say would be the weight of a person $5\frac{1}{2}$ feet tall?

14. The area of the bottom of the foot in persons of normal build varies approximately as the cube of their height. If a man 6 feet tall had a foot 12 inches long, how long a foot would a person 5 feet tall have? (Area of foot varies as the square of its length.)

15. The seaworthiness of a sailing vessel of a certain design varies as the cube of the length of the boat and inversely as the area of the sail. The area of the sail varies as the square of the boat's length. If a vessel 75 feet long is considered satisfactory in this design, would you say that a vessel 25 feet long would be more or less seaworthy? a vessel 100 feet long?

CHAPTER V

EQUATIONS

52. Identities and equations of condition. An equation is a statement of equality between two equal expressions. Equations are of two kinds:

I. Equations that may be reduced to the form $1 = 1$ by performing the indicated operations are called **identities.**

Thus $$4 + 2 = 3 + 3$$
and $$a - b = (3\,a - 2\,b) - (2\,a - b)$$
are equations of this type. In identities the sign $=$ is often replaced by $\equiv$. It should be noted that *identities are true whatever numeric values the letters may have.*

II. Equations that cannot be reduced to the form $1 = 1$, but which are true only when some of the letters have particular values, are called **equations of condition** or simply **equations.**

Thus $x = 2$ cannot be further simplified, and is true only when x has the value 2. Also $x = 2\,a$ is true only when x has the value $2\,a$ or a has the value $\frac{x}{2}$. If in this equation x is replaced by $2\,a$, the equation of condition becomes an identity.

53. Solution of equations. Any number or expression which, on being substituted for a letter in an equation, reduces it to an identity, is said to **satisfy** the equation.

Thus the number 5 satisfies the equation $x^2 - 24 = 1$. The number 3 satisfies the equation $(x - 3)(x + 4) = 0$.

The process of finding values that satisfy an equation is called **solving** the equation. The development of methods for the solutions of various forms of equations is the most important question which algebra considers.

In any equation which involves two or more letters, it is necessary to know which is the one whose value is sought, that is, the letter with respect to which we are expected to solve the equation. The letter whose value is desired is called the **unknown** or sometimes the **variable.**

Values which, when substituted for the unknown, satisfy the equation are called **roots** of the equation.

When only one letter occurs in an equation, the root is a number. When letters other than the unknown occur, the root is usually expressed in terms of those letters.

The practical solution of equations depends on the following

Axiom. *If equal numbers be added to, subtracted from, multiplied by, or divided by equal numbers, the results are equal.*

In the application of this axiom to equations, we exclude, as always, division by zero. In multiplying or dividing both sides of an equation by a literal expression which involves the unknown, it should be observed that the resulting equation may have solutions which the original equation did not possess.

54. Linear equations in one unknown. An equation in which the unknown occurs only to the first degree is called a **linear equation.** To solve a linear equation in one unknown we apply the following

Rule. *Apply the axiom in section 53 to obtain an equation in which the unknown is alone on the left-hand side of the equation. The right-hand side is the desired solution.*

Check. *Substitute the solution in the original equation and reduce to the identity* $1 = 1$.

Since the result of adding two numbers is a definite number, and the same is true for the other operations used in finding the root of a linear equation, it appears that every linear equation in one unknown has one and only one root.

When both sides of an equation between fractions have a common denominator, the numerators are equal to each other. This appears from multiplying both sides of the equation by the common denominator and then canceling it from both fractions.

EXERCISES

Solve and check:

1. $\frac{9-x}{3} - \frac{3x+2}{7} = \frac{5x}{2} - \frac{99}{14}.$

Solution. Multiplying both sides of the equation by 42,

$$126 - 14x - 18x - 12 = 105x - 297.$$

Collecting terms containing x,

$$-105x - 14x - 18x = 12 - 126 - 297.$$

$$-137x = -411.$$

$$x = 3.$$

Check. $\frac{9-3}{3} - \frac{9+2}{7} = \frac{15}{2} - \frac{99}{14} = \frac{3}{7}.$

2. $\frac{ax - a^2 + bx - ab + a - b}{a-b} = \frac{a^2 + 2ab + b^2}{(a-b)^2 + 4ab}.$

Solution. Multiplying by $(a - b)$,

$$ax - a^2 + bx - ab + a - b = \frac{(a+b)^2(a-b)}{(a+b)^2}.$$

$$ax - a^2 + bx - ab + a - b = a - b.$$

Collecting terms containing x,

$$(a+b)x = a^2 + ab.$$

$$x = a.$$

Check. $\frac{a \cdot a - a^2 + b \cdot a - ab + a - b}{a-b} = \frac{a^2 + 2ab + b^2}{(a-b)^2 + 4ab} = 1.$

3. $3x - 5 + 2x = 8x + 2 - 7x.$
4. $2(x+3) + 5(2-x) = 2 + 4x.$
5. $3x + 2(x-1) = 8 + 2x - 5(x+2).$
6. $(x-3)(x+7) = (x+5)(x+1).$
7. $(a+b)x = a + bx.$
8. $m(x^2 + x) = mx^2 - 5.$
9. $(x+b)(x-a) = (x+2)(x-1).$
10. $\frac{2}{3}x - \frac{1}{7} + \frac{1}{2}x - \frac{32}{21} = 2x - \frac{5}{2}x.$
11. $(a+bx)(b+bx) = (bx-a)(bx+a).$
12. $ax - b + cx = 2x - 5 + bx.$
13. $(2+ax)^2 + (3+bx)^2 = (b^2+a^2)(x^2+1).$

14. $10x + \frac{9}{7}(2x + 3) - \frac{2}{3}(5x + 2) = 2(5x + 1) - 1.$

15. $a(x + b)(x + a) = (ax - 2)(x - b).$

16. $\frac{7ax}{2} - \frac{5x}{3} + \frac{2x}{5a} = \frac{a}{2}[(3a - x) + 8(a + x)].$

17. $a + 2b - a(3 - x) + 5x = 7 + 2ax - 9(2 + x).$

18. $(m + x)(m - n) = 3(m + n)(m - x).$

19. $7x - 3 = \frac{1}{2}x + \frac{1}{3}x + \frac{1}{5}x + \frac{1}{7}x + \frac{1}{9}x.$

20. $a(3a - x) + 2x(3 + a) = \frac{5}{2}(2a + ax) - \frac{a}{3}(x + 1).$

21. $(a + b)(a + c + x) + (a - b)(a + c - x) = 2ab + 2bc.$

22. $1.35x - 9.3x + 2.7 + 8.4x - .01x = 1 + .99x - .25.$

23. $3.2\left(\frac{x}{2} - \frac{3}{8}\right) - 8.3x = 9.72 + 5\frac{2}{7}(.25 + .3x).$

24. $x - \frac{1}{2} = \frac{1}{\frac{1}{2} + \frac{1}{\frac{1}{2}}}.$

25. $a + \frac{1}{a} = \frac{1}{x + \frac{1}{a}}.$

26. $\frac{x + a}{x - c} = \frac{x - b}{x + b}.$

27. $\frac{\frac{5}{6}(x + 2)}{\frac{7}{8}(2x - 1)} = \frac{1}{2}.$

28. $\frac{\frac{1}{2}(x + a)}{\frac{2}{3}(x - b)} = \frac{a}{b}.$

29. $\frac{7}{x} + \frac{1}{2} = \frac{15}{x} - \frac{7}{2}.$

30. $\frac{(3x + 2)(2x - 5)}{(6x - 1)(5x + 2)} = \frac{1}{5}.$

31. $\frac{x + 2}{x - 3} + \frac{2}{3} = \frac{x - 3}{x} + \frac{1}{2} + \frac{1}{6}.$

32. $\frac{mx + n}{m + n} = \frac{px - r}{p - r}.$

33. $\frac{9}{10} \cdot \frac{4x - 3}{2x - 1} = 2\left(\frac{9x + 2}{10x - 3}\right).$

34. $\frac{ax}{d} + \frac{bx}{m} + \frac{cx}{n} = p.$

35. $\frac{x + a}{b} - \frac{b}{a} - \frac{a}{b} + \frac{b - x}{a} = 0.$

36. $\frac{2x + a}{a} + m = x + a.$

37. $\frac{ax + b}{bx + a} + \frac{ax - 1}{bx + 2} = \frac{2a}{b}.$

38. $\frac{(2a + x)(3a - 2x)}{(x + 2a)(a - x)} = 2.$

39. $\frac{a + x}{b} + \frac{c - x}{d} - \frac{x - 2}{3} = 0.$

40. $\frac{2a(x + c)}{2d} - \frac{7a(bx - c)}{3d} = \frac{7}{d}.$

41. $\frac{x + p}{2x - m} + \frac{x + n}{x + r} = 3 - \frac{3x^2}{(x + r)(2x - m)}.$

42. $\frac{ax}{c} + \frac{2\,bx}{a} + \frac{2\,ac}{a+b} = \frac{(a-b)^2x}{ab}.$

43. $7\frac{1}{2}\,x + \frac{2}{3}\,x - \frac{9}{10} - \frac{8}{7}\,x = 0.$

44. $\frac{1}{a+b} + \frac{a+b}{x} = \frac{1}{a-b} + \frac{a-b}{x}.$

45. $3(x+1) + \frac{x+1}{x-1} = 3\,x + \frac{5(x+3)}{2(x-1)}.$

46. $\frac{mn-x}{p} - \frac{ab+2\,x}{c} = \frac{3\,am-nx}{m}.$

47. $\frac{ab^2-x}{b} + \frac{bc^2-x}{c} + \frac{a^2c-x}{a} = 0.$

48. $\frac{x}{3\,x^2-4\,x} + \frac{x}{x-2} - \frac{x}{5\,x^2+3\,x} = \frac{x}{x-2}.$

49. $\frac{3\,x-a}{a} + \frac{2\,x+b}{b} - \frac{b}{a}(x+2) = \frac{7(x+a)}{a}.$

50. $\frac{2\,x+3}{3(x+1)} - \frac{8\,x+2}{5(x-1)} + \frac{14\,x^2+2\,x-5}{15\,x^2-15} = 0.$

51. $\frac{2\,x+1}{x+2} + \frac{3\,x+1}{x-1} + \frac{2\,x-10}{x^2+x-2} = 5.$

52. $\frac{7\,b(x+a)}{c} - \frac{3\,a(x-c)}{b} + \frac{2\,c(x-b)}{a} = 0.$

53. $\frac{2\,x-15}{3} + \frac{7\,x+2}{6} - \frac{3\,x-1}{2} = 15.$

54. $\frac{x+a}{bx} - \frac{2\,x-3\,b}{ax} = \frac{2\,a}{x} + \frac{2}{a} - \frac{5}{b}.$

55. $\frac{2\,x-3\,a}{5} + \frac{3\,x-4\,a}{7} = 7\,x + 2\,a.$

56. $\frac{4-2\,x}{3} - \frac{2}{3\,x-1.5} = \frac{3\,x}{2\,x-1} - \frac{2\,x^2}{3(x-.5)}.$

57. $a - \frac{3}{2(x+a)} = \frac{2}{3(x+a)} - \frac{5}{6(x+a)}.$

58. $\frac{abx-cd}{a} + \frac{bcx-df}{b} - \frac{dx-ax}{c} = 0.$

59. $\frac{ax-bc}{ab} - \frac{bx-ac}{c^2} = \frac{cx-b^2}{bc} + \frac{a-x}{c} + \frac{a-x}{a}.$

60. $\frac{x+a}{a} + \frac{x-b}{-b} - \frac{x-a}{-a} = 2\,x.$

61. $\frac{b^2 - 9a^2}{9a^2 - x^2} + \frac{b + x}{x + 3a} = \frac{b - x}{3a - x}.$

62. $\frac{b}{x - b} - \frac{a}{x - a} = \frac{b^2 - a^2}{a(x - a)}.$

63. $\frac{7x - 1}{3} - \frac{2}{5}\left(\frac{x + 2}{3}\right) + \frac{2x - 9}{8} = \frac{17x}{24}.$

64. $\frac{a(x + 1)}{b} + \frac{c(2x - 3)}{d} = \frac{b(2x - 1)}{c}.$

65. $\frac{b(x + 2)}{a} + \frac{b^2(x - 1)}{a^2} - \frac{a(x - 1)}{b} = \frac{a^2(x + 2)}{b^2}.$

66. $\frac{ac}{b(am - bm)} - \frac{x(m + n)^2}{mb} - \frac{n^2x}{bn} = \frac{c}{m(a - b)} - \frac{3nx + b}{b} - 1.$

67. $\left(\frac{a^2}{b} + b\right)(x - a) + \frac{(a + b)(x - b)}{a}(a - b) = 4a^2 + \frac{2ab^2}{b} + 2ax.$

68. $\frac{ax - b}{mx - p} - \frac{an - cm}{mn} - \frac{x(bn + dm) + (bp + dq)}{(mx - p)(nx - q)} = \frac{d + cx}{nx - q}.$

69. $\frac{2a + bx}{3c} + (2ax - 1)\left(\frac{a^2 - 2a}{ab} + 1\right) = \frac{x + 1}{2a} - 3.$

70. $\frac{2x - 5}{3x} + \frac{2ax - 6b}{ax} - \frac{3x + 7b}{bx} = \frac{7a + b}{3ab} - \frac{2b - 3a}{ab}.$

55. Solution of problems. The first essential step in solving a problem by algebra is the expression of the conditions of the problem by means of algebraic symbols. This is, in fact, nothing else than a translation of the problem from the English language into the language of algebra. The translation should be made as close as possible, clause by clause in most cases.

EXAMPLE

What number subtracted from $3\frac{1}{2}$ times itself leaves 20?

Solution. "What number" is translated by x. "Subtracted from $3\frac{1}{2}$ times itself" is translated by $3\frac{1}{2}x - x$, and "leaves 20" is translated by $= 20$. Hence the whole equation becomes, in the language of symbols,

$$3\tfrac{1}{2}x - x = 20.$$

This equation may be solved and checked by the method given above.

PROBLEMS

1. Find a number such that the sum of its half and its fifth part is 35.

2. What number increased by 7 times itself and again by half itself equals 15?

3. Find the number the sum of whose fifth, fourth, and third parts is 51 greater than its half.

4. What number is 12 greater than its third part?

5. In 15 years a boy will be $\frac{2}{3}$ as old as his brother, whose present age is 30. What is the boy's age?

6. A man is 5 years less than 4 times as old as his son. Find the age of the son if the man's age is 35 years.

7. A man had a can of gasoline. He poured 4 gallons more than $\frac{1}{7}$ of the gasoline into the tank of his car and then found that the can was half full of gasoline. How much gasoline had he in the can originally?

8. If a boy were 3 months more than $\frac{3}{5}$ his present age, he would be $6\frac{1}{2}$ years old. How old is he?

9. A boy walked from his home $\frac{2}{3}$ of the distance to town and stole a ride for $\frac{3}{4}$ of the remaining distance. He then had $\frac{1}{10}$ of a mile left to go. How far from town did he live?

10. A man is 1 inch less than 50 % taller than his son. If the father is 6 feet 2 inches tall, find the height of the son.

11. A man kept $\frac{1}{6}$ of his money in his checking account, $\frac{1}{2}$ in first mortgages, and $\frac{1}{5}$ in the savings bank. If he had $10,000 invested in his business, how much money did he have in all?

12. A man takes a walking trip, traveling on successive days $\frac{1}{4}$, $\frac{1}{7}$, $\frac{1}{3}$, and $\frac{1}{8}$ of the total distance, while on the last day he travels 10 miles. Find the length of the trip.

13. Divide 190 into two parts so that one part is $\frac{8}{11}$ of the other part.

14. The sum of two numbers is 56 and their difference is 18. Find the numbers.

15. A boy is 1 year more than twice as old as his brother. The two boys together are 10 years older than their sister, who is 3 years younger than the older boy. Find the age of each.

16. The sum of two numbers is 555. The sum of $\frac{1}{5}$ of one and $\frac{5}{36}$ of the other is 100. Find the numbers.

17. The sum of two numbers is s and their difference is d. Find the numbers.

18. If city A had 12,500 more inhabitants, it would be 3 times as large as city B. City C, of 300,000 inhabitants, has 37,500 less than A and B together. Find the size of A and of B.

19. Two motorists are 100 miles apart. If they travel toward each other at speeds of 20 miles per hour and 25 miles per hour, respectively, how long will it be before they meet?

20. A freight train with an average speed of 25 miles per hour leaves a certain point 3 hours before an express train traveling in the same direction with a speed of 45 miles per hour. How long before the express train overtakes the freight train?

21. Two men, A and B, can cart 50 tons of gravel per day and 45 tons per day, respectively. If B starts work 2 days after A, how long will it take them to haul 200 tons of gravel?

22. One automobile travels 32 miles in 65 minutes, whereas a second automobile travels 27 miles in 45 minutes. How long will it be before the second automobile will have traveled 15 miles farther than the first?

23. The sum of two numbers is 25, and the difference of their squares is 115. Find the numbers.

24. If a certain college had 3 times as many students, it would then have as many more than 1000 students as it now has less than that number. How many students has it now?

25. A man 42 years of age has a son of 15. How long ago was the son $\frac{1}{3}$ as old as his father was?

26. If 12 eggs cost as much less than 81 cents as 15 eggs cost more than 81 cents, how much do the eggs cost apiece?

27. The denominator of a fraction is 3 greater than the numerator. If 6 is subtracted from both the numerator and the denominator, the fraction is equal to $\frac{3}{4}$. Find the fraction.

28. The sum of four numbers is 48. The second number is 4 greater than the first number. The third is 4 greater than the second, and also 4 less than the fourth. Find the numbers.

29. In a certain gathering there were 34 people. There were $\frac{4}{5}$ as many men as women, and 1 more than twice as many women as children. How many were there of each?

30. A boy has $\frac{3}{5}$ as many sisters as brothers. His sister has 3 times as many brothers as sisters. How many boys and girls are there in the family?

31. A man has 8 gallons of 5% alcohol solution in the radiator of his automobile. How much 80% alcohol must he add to make a solution containing 15% alcohol?

32. A mason can get an unlimited amount of gravel composed of 20% sand and 80% small stones. How much sand must he add to 100 cubic feet of gravel to make a mixture containing 30% sand?

33. A train scheduled to make a certain run in 2 hours made the same run in 1 hour and 40 minutes when the average speed was increased 8 miles per hour. Find the length of the run.

34. At what time between 7 and 8 o'clock are the hands of a watch in a straight line?

35. At what time between 4 and 5 o'clock are the hands of a clock at right angles?

36. An oil dealer bought gasoline for 18 cents a gallon and was compelled to sell at 17 cents a gallon. If 2% of his gasoline evaporated in the meantime, and he lost \$13.40 on the transaction, find the amount of gasoline which he purchased.

37. A tank can be filled by a pipe in 20 minutes, and emptied by a second pipe in 25 minutes. With both pipes open how long will it take to fill the tank?

38. An airplane can cover a distance in 3 hours with the wind, but can only get back $\frac{4}{5}$ of the way in the same time. If the speed of the plane in still weather is 90 miles per hour, find the speed of the wind.

39. A swimming pool has 3 drains which can empty it in 20 minutes, 45 minutes, and 60 minutes, respectively. If all the drains are left open, how long will it take to empty the pool?

40. A certain piece of work can be done by A in 2 days' less time than B requires, while B can do it in half the time required by C. If B and C together can do it in $3\frac{1}{3}$ days, how long will it take all three working together to do the work?

56. Linear equations in two unknowns. A linear equation in one unknown has one and only one solution, as we have already seen (section 54). On the other hand, an equation of the first degree in two unknowns has many solutions.

For example, $2x + 3y = 12$

is satisfied by innumerable pairs of numbers which may be substituted for x and y. For, transposing the term in y, we find that

$$x = \frac{12 - 3y}{2},$$

from which it appears that when y has any particular numeric value the equation becomes a linear equation in x alone, and hence has a solution. Thus when $y = 2$, $x = 3$, and this pair of values is a solution of the equation. Similarly, $y = 6$ and $x = -3$ also satisfy the equation.

At the same time there are countless pairs of values of x and y which do not satisfy any given equation.

For example, the values $x = 2$, $y = 1$, or $x = -1$, $y = 3$ do not satisfy the equation above.

57. Graphical solution of a linear equation in two unknowns. The construction of the graph of a single linear equation in two unknowns (or of a linear system in two unknowns) depends on several assumptions and definitions. These are:

I. That we have at right angles to each other two lines: $X'OX$, called the **x-axis**, and $Y'OY$, called the **y-axis**.

II. That we have a unit of distance on these two lines.

Thus the number 2 will correspond to a distance of twice the unit, the number $4\frac{1}{2}$ will correspond to a distance $4\frac{1}{2}$ times the unit, etc.

III. That the distance (measured parallel to the x-axis) from the y-axis to any point on the paper be the **x-distance** (or **abscissa**) of the point, and the distance (measured parallel to the y-axis) from the x-axis to the point be the **y-distance** (or **ordinate**) of the point.

IV. That the x-distance of a point to the *right* of the y-axis be represented by a *positive number*, and the x-distance of a point to the *left* by a *negative number*; also that the y-distance of a point *above* the x-axis be represented by a *positive number*, and the y-distance of a point *below* the x-axis by a *negative number*. Briefly, distances measured from an axis to the *right* or *upward* are *positive*, and those measured to the *left* or *downward* are *negative*.

V. That every point in the surface of the paper corresponds to a pair of numbers, one or both of which may be positive or negative, integral or fractional.

VI. That in stating the pair of numbers corresponding to the location of a given point, the x-distance be stated first and the y-distance second.

Thus the point (2, 3) is the point whose x-distance is 2, and whose y-distance is 3.

The point of intersection of the axes is called the **origin**.

The values of the x-distance and the y-distance are often called the **coördinates** of the point.

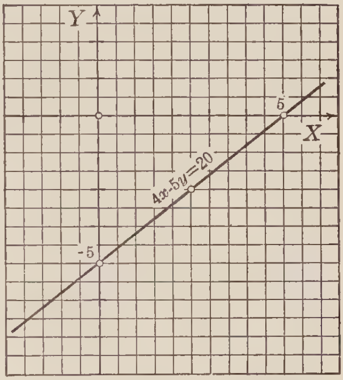
The relation between an equation and its graph may be stated as follows:

The equation of a line is satisfied by the values of the x-distance and the y-distance of any point on that line.

Any point the values of whose x-distance and whose y-distance satisfy the equation is on the graph of the equation.

The graph of a linear equation in two unknowns is a straight line. Therefore it is necessary, in constructing the graph of such an equation, to locate only two points whose coördinates satisfy the equation, and then draw through the two points a straight line. It is usually most convenient to locate the two points where the line cuts the axes. If, however, these two points are very close together, the direction of the line will not be accurately determined by them. This difficulty can be avoided by selecting two points at a greater distance apart.

If one considers the simplest kind of equation in two variables, such as $x + y = 6$, one sees immediately that there are countless pairs of roots, since there are countless numbers whose sum is 6. There will be, therefore, countless points on the graph of $x + y = 6$. It is just as clear that there are countless pairs of numbers that do not add up to 6. Hence there will be countless points which do not lie on the graph of $x + y = 6$. If one selected two numbers at random, one would say there was only a slight chance that they would add up to 6. It is a striking result of the type of graphical display that is defined in the preceding paragraphs that all the points whose coördinates satisfy the equation do not lie in a little bunch, nor are they scattered loosely over the plane, but lie on a straight line. Thus the simplest equation is represented by the simplest type of geometric figure.



EXAMPLE

Graph $4x - 5y = 20$.

Solution. Substituting zero for x, we obtain $y = -4$.

Similarly, when $y = 0$, $x = 5$, and when $y = -2$, $x = 2\frac{1}{2}$.

Plotting the points found above, we obtain the graph here shown.

58. Solution of a pair of equations. If the values that a pair of numbers, x and y, may assume are restricted not merely by one linear equation, but by two, there is usually one and only one solution. As an instance of this we have the following

EXAMPLE

Solve graphically

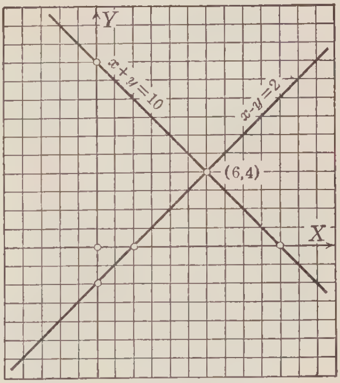
$$x + y = 10, \qquad (1)$$
$$x - y = 2. \qquad (2)$$

Solution. For (1)

x	0	10
y	10	0

For (2)

x	0	2
y	-2	0



Of course, there are countless pairs of numbers whose sum is 10, that is, for which the equation $x + y = 10$ is valid. But if one imposes the further restriction that the difference of the two numbers must be 2, that is, that $x - y = 2$, it appears that the numbers 6 and 4 make up the only pair of values which satisfy both conditions. This fact appears in the preceding graphical solution.

59. Dependent equations and systems. If two equations are reducible to the same form, they are said to be **dependent.** If they are not reducible to the same form, they are **independent.**

Thus
$$6x - 8y - 4 = 0,$$
and
$$3x - 4y = 2$$
are dependent, since the first is readily reduced to the second by transposing and dividing by 2. They are, in fact, essentially the same equation. On the other hand,
$$x - 4y = 2,$$
and
$$3x - 4y = 2$$
are not reducible to the same form and are independent. Since dependent equations are identical except for the arrangement of terms and some constant factor, all their solutions are common to each other.

In a graph dependent equations are represented by the same line, while independent equations always have different graphs.

We may summarize the above facts as follows:

Two equations,
$$ax + by + c = 0$$
and
$$a'x + b'y + c' = 0,$$
are dependent when and only when
$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

Independent equations in more than one unknown which have a common solution are called **simultaneous** equations.

Two systems of simultaneous equations which are satisfied by the same pair (or pairs) of values of x and y and only these are called **equivalent** systems.

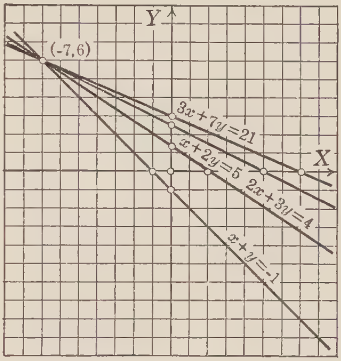
EXAMPLE

Solve graphically

$$3x + 7y = 21, \quad x + y = -1; \qquad \text{and} \qquad x + 2y = 5, \quad 2x + 3y = 4.$$

Solution. The two systems of equations given above are equivalent, the common solution being $x = -7$, $y = 6$.

In the graph of the above systems (and, in general, for all equivalent systems) the lines for all the equations pass through the same point. Thus



$3x + 7y = 21$

x	0	7
y	3	0

$x + y = -1$

x	0	-1
y	-1	0

$x + 2y = 5$

x	0	5
y	$2\frac{1}{2}$	0

$2x + 3y = 4$

x	0	2
y	$\frac{4}{3}$	0

60. Solution of simultaneous linear equations. For the solution of a pair of simultaneous linear equations we have the

Rule. *Multiply each of the equations by some number such that the coefficients of one of the unknowns in the resulting pair of equations are identical.*

Subtract one equation from the other and solve the resulting simple equation in one unknown.

Find the value of the other unknown by substituting the value just found in one of the original equations.

Check the result by substituting the values found for both unknowns in the other original equation.

EXAMPLE

Solve and graph $3x + 7y = 1,$ (1)

$$x + y = -1. \quad (2)$$

Solution. Multiplying equation (2) by 3, and subtracting (2) from (1),

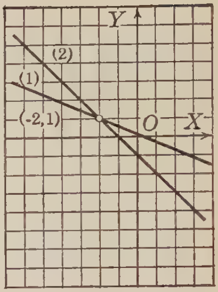
$$\begin{array}{r} 3x + 7y = 1 \\ \underline{3x + 3y = -3} \\ 4y = 4 \\ y = 1. \end{array}$$

Substituting in (2), $x + 1 = -1.$

$$x = -2.$$

Check. Substituting in (1),

$$3 \cdot (-2) + 7 \cdot 1 = -6 + 7 = 1.$$



61. Incompatible equations. Equations in more than one unknown that do not have a common solution are called **incompatible.**

Theorem. *The independent equations*

$$ax + by = c, \quad (1)$$

$$a'x + b'y = c' \quad (2)$$

are incompatible when and only when $ab' - a'b = 0.$

Apply the rule of section 60 to find the solution of these equations.

We obtain $aa'x + a'by = a'c,$

$$aa'x + ab'y = ac'.$$

Subtracting, $(ab' - a'b)y = ac' - a'c.$

If $(ab' - a'b)$ were not equal to zero, we would get a value of y. But if $(ab' - a'b) = 0$, we can get no value for y, since division by zero is ruled out (section 7), and no solution of (1) and (2) exists.

It is observed that the condition $ab' - a'b = 0$ is the same as the condition $\frac{a'}{b'} = \frac{a}{b}$; that is, the condition that two linear equations (1) and (2) have no solution is that the coefficients of the terms containing the unknowns are proportional. We have already seen that if this proportion equals the ratio of the constant terms, the equations are equivalent.

EXAMPLE

Solve and graph

$$3x + 7y = 1, \qquad (1)$$

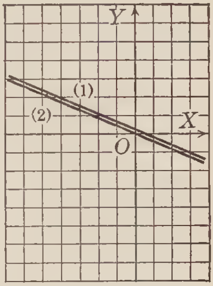
$$6x + 14y = 1. \qquad (2)$$

Solution. Multiplying (1) by 2, and subtracting (2) from (1),

$$\begin{array}{r} 6x + 14y = 2 \\ \underline{6x + 14y = 1} \\ 0 = 1 \end{array}$$

which is absurd. Thus no solution exists.

This situation is shown in the adjacent graph.



62. Résumé. We observe that pairs of equations of the form

$$ax + by + c = 0,$$

$$a'x + b'x + c' = 0$$

fall into three classes:

(*a*) *Dependent equations*, which have innumerable common solutions.

Then $$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}. \qquad (1)$$

The graphs of dependent linear equations in two unknowns are identical lines.

(*b*) *Incompatible equations*, which have no common solution.

Then $$ab' - a'b = 0$$

but (1) is not true, that is, $bc' - b'c \neq 0$.

The graphs of incompatible linear equations in two unknowns are two parallel lines.

(*c*) *Simultaneous equations*, which have one and only one pair of solutions.

Then $$ab' - a'b \neq 0.$$

The graphs of two simultaneous equations in two unknowns are two intersecting lines.

EXERCISES

Solve and check the following. Graph as directed by the teacher.

1. $3x + 2y = 8$,
$5x + 3y = 13$.

2. $7x - 5y = 18$,
$\frac{3}{2}x + 2y = 10$.

3. $4x - 2y = 2$,
$3x - y = 0$.

4. $3x - 9y = 3$,
$2x - 5y = 2$.

5. $3x - \frac{5}{4}y = 16$,
$4x - \frac{5}{3}y = 24$.

6. $\frac{2}{3}x = \frac{1}{7}y + 13$,
$\frac{3}{7}x = \frac{5}{14}y + y - \frac{1}{2}$.

7. $4x - 9y = -10$,
$1.5x + 3.2y = 9.4$.

8. $x = 7y$,
$\frac{1}{2}x - 7y = -7$.

9. $4x + 3y - 6 = 0$,
$3x - 5y - 19 = 0$.

10. $2x + ay = b$,
$3x - by = c$.

11. $x + y = 3a + b$,
$x - y = a - 3b$.

12. $4x - 5y = 9a$,
$5x - 4y = 9b$.

13. $\frac{2}{3}x + \frac{1}{2}(y + 2) = 5$,
$\frac{1}{6}x - \frac{1}{8}(y + 8) = -1$.

14. $2.75x - 2.25y = -y + .25x$,
$8.375y + 1.25x = 9y$.

15. $7x + 5y = a^2 + ab + b^2$,
$7y + 5x = a^2 - ab + b^2$.

16. $\frac{1}{x} + \frac{1}{y} = 2$,
$\frac{1}{x} - \frac{1}{y} = 0$.

HINT. Solve without clearing of fractions.

17. $\frac{3}{x} + \frac{5}{y} = \frac{5}{6}$,
$\frac{2}{x} - \frac{3}{y} = \frac{2}{15}$.

18. $\frac{x - a}{y - b} = \frac{c}{d}$,
$x + y = a + b$.

19. $\frac{2x + 3}{y - 2} = 7$,
$\frac{3x + 2}{2 - y} = -8$.

20. $\frac{x + y - 1}{2x - y + 2} = \frac{1}{3}$,
$\frac{2x + y - 2}{x + y} = \frac{1}{2}$.

21. $\frac{x}{m} + \frac{y}{n} = p$,
$\frac{x}{a} + \frac{y}{b} = c$.

22. $\frac{2}{x + 2y} = \frac{4}{4x + 3y}$,
$\frac{4}{4x - y} = \frac{2}{7x - 3y}$.

23. $\frac{x + 1}{y + 1} = \frac{a + b - c}{a - b - c}$,
$\frac{x - 1}{y - 1} = \frac{a - b + c}{a + b + c}$.

24. $$\frac{.8\,x+.2\,y+.5}{12\,x-13\,y-14}=-.1,$$
$$\frac{.7\,x+.1\,y+.4}{7\,x+2\,y-3}=.2.$$

25. $$x+y=\frac{a^2+4\,ab}{2\,a},$$
$$x-y=\frac{a^2-b^2}{a^2+2\,ab+b^2}.$$

26. $$\frac{x}{a-b}+\frac{y}{a+b}=\frac{1}{a-b},$$
$$\frac{x}{a-b}-\frac{y}{a+b}=\frac{1}{b-a}.$$

27. $$\frac{x}{a-b}-\frac{y}{a-c}=\frac{a-b}{a-c},$$
$$\frac{a-x}{b}+\frac{a-y}{c}=\frac{a-b+2\,c}{c}.$$

28. $$\sqrt{x}=2+\sqrt{y},$$
$$\sqrt{y}=7-2\sqrt{x}.$$

29. $$3\,x-\sqrt{y}=2,$$
$$3\sqrt{y}+2\,x=5.$$

30. $$2\sqrt{x}-\sqrt{y}=1,$$
$$\sqrt{x}-2\sqrt{y}=-1.$$

31. $$x\sqrt{a}-y\sqrt{b}=a+b,$$
$$x+y=2\sqrt{a}.$$

32. $$x\sqrt{3}+y\sqrt{5}=8,$$
$$x\sqrt{5}-y\sqrt{3}=0.$$

33. $$2\sqrt{x+2}-2\sqrt{y-2}=2,$$
$$7\sqrt{x+2}+3\sqrt{y-2}=27.$$

34. $$\frac{2}{\sqrt{x}}+\frac{4}{\sqrt{y}}=6,$$
$$\frac{3}{\sqrt{x}}-\frac{5}{\sqrt{y}}=-2.$$

35. $$\frac{27}{\sqrt{x-1}}+\frac{6}{\sqrt{y-2}}=25,$$
$$\frac{3}{\sqrt{x-1}}-\frac{21}{\sqrt{y-2}}=10.$$

36. $$(a-b)x+y=a^2,$$
$$(a-b)(x+ay+by)=(a^2-b^2)(b^2+1).$$

37. $$\frac{x+1}{3}-\frac{y-2}{4}=\frac{(x+y)}{-1},$$
$$3(x+2)+7(y-3)=3\,y+x+1.$$

63. Solution of problems involving two unknowns. The same principle of translating the problem into algebraic symbols should be followed here as in the solution of problems leading to simple equations (section 55).

PROBLEMS

1. The difference between two numbers is 5, and their sum is $12\frac{1}{3}$. Find the numbers.

2. Find the two numbers whose sum is s and whose difference is $-s$.

3. Two men divide $5000 between them. If one man had received $1000 more and the other had received $500 less, they would each have received equal amounts. How much did each man receive?

4. A man stands between two houses that are 45 feet apart. If he were 3 times as far from the first house and 15 times as far from the second house, he would then be twice as far from the second as from the first. Find his distance from each house.

5. There are two groups of boys on the street corner. One boy leaves the first group and joins the second. The groups are then equal in size. If one boy had left the second group and joined the first, the first group would then have been 3 times as large as the second. Find the original size of each group.

6. A man owns 40 shares of stock. One kind of stock pays $10.75 annually per share, while the other kind pays $4.50 per share annually. If his total income from this source is $242.50 per year, find how many shares of each stock the man owns.

7. An automobile cost 40 times as much as one of its tires and $1800 more than all four of its tires. Find the cost of the car and each of the tires.

8. A man considering the purchase of an automobile may buy a certain make of car and spend $\frac{1}{20}$ of the price for accessories, or he may purchase another make of car costing half as much as the first and spend $\frac{3}{10}$ of the price of that car for accessories. If the sum of the amounts he would spend for accessories on each car is $150, find how much he must spend to fit out each car.

9. What fraction becomes $\frac{2}{3}$ when 2 is subtracted from the numerator and denominator, and becomes its own reciprocal (section 43) when 1 is added to the numerator and subtracted from the denominator?

10. A man may purchase either one of two automobiles, a small car which will cost him $\frac{1}{2}$ its price each year to own and $300 a year to run, or a large car which will cost him $\frac{1}{8}$ its price each year to own and $500 a year to run. If the large car had cost $100 more, the annual expenses of the two cars would have been equal. Find the cost of each car if the large car costs 2.3 times as much as the small car.

11. A man invests a sum of money in stocks. The first year they yield him \$200. The second year the interest rate rose $\frac{2}{5}$% and the amount of his income rose \$20. How much money did he invest?

12. A man has his money invested, part at 4% and the remainder at 7%. The income is then \$145. If he had received $\frac{1}{2}$% less interest, his income would have been \$132.50. How much money was invested at each rate of interest?

13. When a two-digit number is added to 27 the order of the digits is reversed. When 14 is subtracted from the number the sum of the digits equals 2. Find the number.

14. A man has two sums of money at interest, the first at 4% and the second at $4\frac{1}{2}$%. The yield from the first is half as much as the yield from the second. If the amounts were interchanged, the income would be \$1045. Find the amount invested at each rate of interest.

15. An imitation gold nugget in a museum weighs 25 pounds. It is made of lead covered with a gold shell. Under water its weight is $22\frac{4}{5}$ pounds. If $19\frac{1}{3}$ pounds of gold and $11\frac{1}{3}$ pounds of lead each lose one pound when weighed under water, find the amount of each metal in the nugget.

16. A part of a certain machine is made of iron and lead, and weighs 6 pounds. If its weight under water is $5\frac{1}{4}$ pounds, and if $7\frac{7}{8}$ pounds of iron and $11\frac{1}{3}$ pounds of lead each lose one pound when weighed under water, find the amount of each metal in the part.

17. A group consists of 30 people who have between them \$100. If each man has \$2 and each woman has \$4, find how many men and how many women there are in the group.

18. Two numbers containing the same two digits differ by 9. The greater digit is 3 more than half the smaller digit. Find the two digits.

19. A chemist has two strengths of a certain solution. One part of the first and 4 parts of the second give a solution of 27% strength. If the amount of the two solutions taken is reversed, the strength of the resulting solution is 37%. What is the strength of each of the original solutions?

20. Two men, A and B, start 24 miles apart and walk toward each other. If A starts 1 hour before B starts, they will meet 3 hours after B starts. If B starts 1 hour before A, they will meet $2\frac{6}{7}$ hours after A starts. How fast does each man walk?

64. Solution of linear equations in several unknowns. This process is performed as follows:

Rule. *Eliminate one unknown from the equations, taking them in pairs, thus giving a system of one less equation than at first in one less unknown.*

Continue the process until the value of one unknown is found.

The remaining unknowns may be found by substitution.

Special cases occur, as in the case of two unknowns, where an infinite number of solutions or no solution exists. Where no solution exists one is led to a self-contradictory equation on application of the rule.

A more detailed study of these special cases is found in Chapter XIX.

EXERCISES

Solve and check:

1. $x + y + z = 6,$ (1)
$3x + 2y - 4z = -5,$ (2)
$7x - y - 2z = -1.$ (3)

Solution. $(1) + (3),$ $8x - z = 5.$ (4)
$(2) + 2 \cdot (3),$ $17x - 8z = -7.$ (5)
$8 \cdot (4) - (5),$ $47x = 47.$
$x = 1.$
In (4), $z = 3.$
In (1), $y = 2.$

Check. In (2), $3 + 4 - 12 = -5.$

2. $x + y = 4,$
$2x - z = 2,$
$y + z = 4.$

3. $x + y = 1,$
$3x + 2y - z = 1,$
$5z - 2y = 3.$

4. $2x - 2y + z = 3,$
$3x - 5y - z = -2,$
$7x - 4y + 2z = 15.$

5. $x + y + 2z = a,$
$y + z + 2x = b,$
$z + x + 2y = c.$

6. $x + 2y - z = 14,$
$3x = y,$
$7x = 3z.$

7. $x - y = 3z - 10,$
$7x + 6y = 19,$
$z + y = 5x.$

8. $1.3x + 2.1y = 13.7,$
$2.7y - .5z = -.1,$
$.7x + .2z = 7.4.$

9. $x + 2y = 0,$
$y + 2z = 3,$
$z + 2u = 7,$
$u + 2x = 7.$

10. $x + y = a,$
$y + z = b,$
$z + u = c,$
$u + x = d.$

11. $\frac{1}{x} - \frac{1}{y} = -1,$
$\frac{3}{y} - \frac{2}{z} = -1,$
$\frac{2}{x} + \frac{3}{z} = 19.$

12. $\frac{xy}{x+y} = \frac{6}{5},$
$\frac{xz}{x+z} = \frac{12}{7},$
$\frac{yz}{y+z} = \frac{4}{3}.$

13. $\frac{xy}{4y - 3x} = 20,$
$\frac{xz}{2x - 3z} = 15,$
$\frac{yz}{4y - 5z} = 12.$

14. $\frac{x+1}{y+1} = \frac{5}{4},$
$\frac{y+3}{z+1} = -6,$
$\frac{z+2}{x+1} = 0.$

15. $2\frac{1}{2}y = x + z - 2,$
$3\frac{1}{3}z = y + x + 4,$
$4\frac{1}{4}x = y + z + 12.$

16. $x + y = 2\frac{2}{3}z + 3,$
$x + z = 4\frac{4}{5}y - 15,$
$y + z = 5\frac{5}{6}x - 27.$

17. $3x + 7y + 2z = 31,$
$x - 3y + z = -5,$
$2x + 3y - 2z = 9.$

18. $x - 7y + 2z = 4,$
$3x - 2y - 2z = -4,$
$7x + 2y + z = 2.$

19. $7x + 2y - 13z = -13,$
$2x + 3y - 2z = -10,$
$8x - 2y - 5z = 22.$

20. $2y + 3z - 4x = 4,$
$7y + z - x = -5,$
$x + y + z = 1.$

21. $(x - 2)(3y - 7) = (3x + 2)(y + 2),$
$(x + 3)(3z + 2) = (3x - 1)(z + 6),$
$(7y - 9)(2x - 3) = (7x + 2)(2y + 1).$

CHAPTER VI

IRRATIONAL NUMBERS AND RADICALS

65. Existence of irrational numbers. We have seen that in order to solve any linear equation or set of linear equations with rational coefficients we need to make use only of the operations of addition, subtraction, multiplication, and division. When, however, we attempt to solve an equation of the second degree, such as $x^2 = 2$, we find that there is no rational number that satisfies it.

Assumption. *A factor of one member of an identity between integers is also a factor of the other member.*

Thus let $2 \cdot a = b$, where a and b are integers. Then, since 2 is a factor of the left-hand member, it must also be contained in b.

Theorem. *No rational number satisfies the equation* $x^2 = 2$.

Suppose the rational number $\frac{a}{b}$ to be a fraction reduced to its lowest terms which satisfies the equation. Then, by the hypothesis,

$$\left(\frac{a}{b}\right)^2 = 2,$$

or

$$a^2 = 2\,b^2. \qquad (1)$$

Hence, by the assumption, 2 is contained in a^2, and consequently in a.

Suppose $\quad a = 2\,a'.$

Then, by (1), $\quad 4\,a'^2 = 2\,b^2,$

or $\quad 2\,a'^2 = b^2;$

that is, 2 must also be contained in b, which contradicts the hypothesis that $\frac{a}{b}$ is a fraction reduced to its lowest terms.

This is equivalent to the statement that the square root of 2 cannot be expressed exactly by a rational fraction.

The fact that the equation $x^2 = 2$ has no rational solution is analogous to the fact, learned from geometry, that the hypotenuse of an isosceles right triangle is incommensurable with the other sides.

66. The practical necessity for irrational numbers. For the practical purposes of the draftsman, the surveyor, or the machinist, the introduction of this irrational number is superfluous, as no measuring scale can be made exact enough to distinguish between a length represented by a rational number and a length which cannot be so represented. As the draftsman does not use a mathematically perfect triangle, but one made of wood or celluloid, it is impossible to find, in the so-called practical world, an instance where the irrational number is indispensable. In fact the irrational number is a *mathematical* necessity, and not a necessity for the laboratory or drafting room in the sense that fractions and negative numbers are. We need irrational numbers because we cannot solve all quadratic equations without them. The *practical* utility of these numbers comes only through the immense gain in mathematical power which they bring.

67. Extraction of square root of polynomials. This process, from which a method of extracting the square root of numbers is immediately deduced, may be performed as follows:

Rule. *Arrange the terms of the polynomial according to the powers of some letter.*

Extract the square root of the first term, write the result as the first term of the root, and subtract its square from the given polynomial.

Divide the first term of the remainder by twice the root already found, and add this quotient to the root and also to the trial divisor, thus forming the complete divisor.

Multiply the complete divisor by the last term of the root and subtract the product from the last remainder.

If terms of the given polynomial still remain, find the next term of the root by dividing the first term of the remainder by twice the first term of the root, form the complete divisor, and proceed as before until the desired number of terms of the root have been found.

EXERCISES

Extract the square root of the following:

1. $4\,a^2 + 4\,ax^2 - 4\,abx + x^4 + b^2x^2 - 2\,bx^3$.

Solution.

$$\begin{array}{ll}
 & \underline{2\,a - bx + x^2 \qquad\qquad\qquad\qquad\qquad} \\
 & 4\,a^2 - 4\,abx + 4\,ax^2 + b^2x^2 - 2\,bx^3 + x^4 \\
 & \underline{4\,a^2 \qquad\qquad\qquad\qquad\qquad\qquad} \\
4\,a & \big| -4\,abx + 4\,ax^2 + b^2x^2 \\
4\,a - bx & \big| \underline{-4\,abx \qquad\qquad + b^2x^2 \qquad\qquad\qquad} \\
4\,a - 2\,bx & \qquad\quad +4\,ax^2 \qquad\quad - 2\,bx^3 + x^4 \\
4\,a - 2\,bx + x^2 & \qquad\quad \underline{+4\,ax^2 \qquad\quad - 2\,bx^3 + x^4}
\end{array}$$

2. $4\,x^4 - 4\,x^3 - 3\,x^2 + 2\,x + 1$.
3. $x^4 + 4\,x^2 + 9 - 4\,x^3 + 6\,x^2 - 12\,x$.
4. $x^4 + 3\,x^2y^2 + y^4 + 2\,x^3y + 2\,xy^3$.
5. $4\,x^4 + 28\,ax^3 + 53\,a^2x^2 + 14\,a^3x + a^4$.
6. $9\,m^2 + 4\,n^2 + 9 + 12\,mn - 18\,m - 12\,n$.
7. $a^2 - 2\,ab + b^2 + x^2 + 2\,ax - 2\,bx$.
8. $1 - a^2 + 25\,a^4 - 6\,a + 30\,a^3$.
9. $x^4 - 2\,x^3 + 7\,x^2 + 9 - 6\,x$.
10. $1 + b$, to four terms.
11. $1 - b$, to four terms.
12. $4 - 3\,b$, to four terms.

68. Extraction of the square root of numbers. A study of the method employed to find the square root of polynomials enables us to state, for the extraction of the square root of numbers, the following

Rule. *Separate the number into periods of two figures each, beginning at the decimal point. Find the greatest number whose square is contained in the left-hand period. This is the first figure of the required root.*

Subtract its square from the first period, and to the remainder annex the next period of the number.

Divide this remainder, omitting the right-hand digit, by twice the root already found, and annex the quotient to both root and divisor, thus forming the complete divisor.

Multiply the complete divisor by the last digit of the root, subtract the result from the dividend, and annex to the remainder the next period for a new dividend.

Double the whole root now found for a new divisor and proceed as before until the desired number of digits have been found.

In applying this rule it often happens that the product of the complete divisor and the last digit of the root is larger than the dividend. In such a case we must diminish the last figure of the root by unity until we obtain a product which is not greater than the dividend.

At any point in the process of extracting the square root of a number before the exact square root is found, the square of the result already obtained is less than the original number. If the last digit of the result is replaced by the next higher one, the square of this number is greater than the original number.

69. Principal roots. There are always two square roots of any number. Thus the number 4 has the two square roots $+2$ and -2. The sign of the root which is intended must be indicated by the sign before the radical sign when this sign is used. As usual, if no sign is written, a $+$ is understood.

EXERCISES

Extract the square root of the following, to the nearest thousandth:

1. 3.

Solution.

$$\begin{array}{r|l}
 & \underline{1.\ 7\ \ 3\ \ 2\ \ 0\ \ 5} \\
 & 3.00'00'00'00'00 \\
 & \underline{1} \\
2\ 7 & 2\ 00 \\
 & \underline{1\ 89} \\
34\ 3 & 11\ 00 \\
 & \underline{10\ 29} \\
346\ 2 & 71\ 00 \\
 & \underline{69\ 24} \\
3464\ 05 & 1\ 76\ 00\ 00 \\
 & \underline{1\ 73\ 20\ 25} \\
 & \quad\quad 2\ 79\ 75
\end{array}$$

1.732. *Ans.*

2. 7569.
3. 1764.
4. 30.25.
5. 201375.
6. 3.14159.
7. 8.73210793.
8. .00032098214.
9. .00001492.
10. 1928.01.
11. 1500.
12. 7.
13. 11.
14. 10.
15. 1.414.
16. 7.56.
17. 1,520,876.
18. 13.
19. 2.
20. 549.
21. 23.

70. Approximation of irrational numbers. In applying the preceding process to obtain the square root of 2 we can never obtain a number whose square is exactly 2, for we have seen that such a number expressed as a rational fraction does not exist. But as we proceed we get a number whose square differs less and less from 2.

Thus $1^2 = 1$, less than 2 by 1.

$1.4^2 = 1.96$, less than 2 by .04.

$1.41^2 = 1.9881$, less than 2 by .0119.

$1.414^2 = 1.999396$, less than 2 by .000604.

Though we cannot say that 1.414 *is* the square root of 2, we may say that 1.414 is the square root of 2 to three places of decimals, meaning that

$$(1.414)^2 < 2 < (1.415)^2.$$

71. Sequences. The exact value of the square root of most numbers, as, for instance, 2, 3, 5, cannot be found exactly in decimal form and so is usually expressed symbolically. By means of the process of extracting the square root, however, we can find a number whose square is as near the given number in value as we may desire. We may, in fact, assert that the succession, or sequence, of numbers obtained by the process of extracting the square root of a number defines the square root of that number. Thus the sequence of numbers 1, 1.4, 1.41, 1.414, · · · defines the square root of 2.

72. Operations on irrational numbers. Just as we defined the laws of operation on fractions and negative numbers (sections 2–10) we should now define the meaning of the sum, difference, product, and quotient of the numbers defined by the sequence of numbers obtained by the square-root process.

To define and explain completely the operations on irrational numbers is beyond the scope of this chapter. It turns out, however, that the number defined by a sequence is the limiting value of the rational numbers that constitute the sequence; that is, it is a value from which every number in the sequence beyond a certain point differs by as little as we please. We may, however, make the following statement regarding the multiplication of irrational numbers: In the sequence defining the square root of 2, namely, (1, 1.4, 1.41, 1.414, $\cdots$) we saw that we could obtain a number very nearly equal to 2 by multiplying 1.414 by itself. In general, *we may multiply numbers defined by sequences by multiplying the elements of these sequences; the new sequence, consisting of the products, defines the product of the original numbers.*

Thus $(1,\ 1.4,\ 1.41,\ 1.414, \cdots)(1,\ 1.4,\ 1.41,\ 1.414, \cdots)$

$$= (1,\ 1.96,\ 1.9881,\ 1.999396, \cdots).$$

The numbers in this sequence approach 2 as a limit, and hence the sequence may be said to represent 2.

73. Notation. We denote the square root of a (where a represents any number or expression) symbolically by $\sqrt{a}$, and assert that

$$\sqrt{a} \cdot \sqrt{a} = (\sqrt{a})^2 = a,$$

or, more generally, $\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$.

Similarly, $$\sqrt{a} \div \sqrt{b} = \sqrt{a \div b}.$$

EXERCISES

1. Form five elements of a sequence defining $\sqrt{2}$.

2. Form four elements of a sequence defining $\sqrt{6}$.

3. Form five elements of a sequence defining $\sqrt{15}$.

4. Form, in accordance with the rule just given, four elements of the sequence $\sqrt{3} \cdot \sqrt{5}$. Compare this result with the elements obtained in Exercise 3.

5. Form the first four elements of the product $\sqrt{2} \cdot \sqrt{7}$, and compare it with the first four elements defining $\sqrt{14}$.

74. Other irrational numbers. The cube root and higher roots of numbers could also be found by processes analogous to the method employed in finding the square root, but as they are almost never used practically, they will not be included here. It should be kept in mind, however, that by these processes sequences of numbers may be derived that define the various roots of numbers precisely as the sequences derived in the preceding paragraphs define the square root of numbers.

The nth root of any expression a is symbolized by $\sqrt[n]{a}$. Here n is sometimes called the *index* of the *radical*. The principle governing the multiplication and division of radicals with any integral index is given by the following

Assumption. *The product (or quotient) of the nth roots of two numbers is equal to the nth root of the product (or quotient) of the numbers.*

Symbolically expressed,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b},$$

or

$$\sqrt[n]{a} \div \sqrt[n]{b} = \sqrt[n]{a \div b}.$$

For a given index the *principal root* of a number is its one real root if it has but one, or its positive root if it has two real roots.

Thus the principal square root of 9 is $+3$, and the principal fourth root of 16 is $+2$, not -2. The square root of -4 or -9 is not a real number; a negative number has no principal square root.

The principal cube root of 8 is 2; that of -27 is -3. The principal fifth root of 32 is $+2$; that of -32 is -2.

Every number has more than one root of given odd index. The number 8, for example, has two other cube roots besides its principal cube root, 2. What they are and how they are obtained will be made clear in the chapter on Complex Numbers, where the consideration of the square roots of negative numbers is taken up.

75. Reduction of a radical to its simplest form. A radical is in its simplest form when the expression under the sign is integral (section 11) and contains no factor raised to a power which equals or exceeds the index of the radical; in other words, when no factor can be removed from under the radical sign and still leave an integral expression under the sign.

Thus the radical $\sqrt{10}$ is in its simplest form, since 10 does not contain a perfect square as a factor. However, $\sqrt{18}$ is not in its simplest form, since the perfect square 9 is a factor of 18, and $\sqrt{18}$ may therefore be written $\sqrt{9} \cdot \sqrt{2}$, or $3\sqrt{2}$.

The simplification of radicals is based on the following general identity, which is an application of the assumption of the preceding section:

$$\sqrt[n]{a^n b} = \sqrt[n]{a^n} \cdot \sqrt[n]{b} = a\sqrt[n]{b}.$$

We may reduce a *quadratic* radical to its simplest form by the following

Rule. *If the expression under the radical sign is fractional, multiply both numerator and denominator by some expression that will make the denominator a perfect square.*

Factor the expression under the radical into two factors, one of which is the greatest square factor which it contains.

Take the square root of the factor that is the perfect square, and express the multiplication of the result by the remaining factor under the radical sign.

If the radical is of the nth index, the denominator must be made a perfect nth power, and any factor that is to be taken from under the radical sign must also be a perfect nth power.

EXERCISES

Reduce to simplest form:

1. $\sqrt{\frac{18}{5}}$.

Solution. $\sqrt{\frac{18}{5}} = \sqrt{\frac{9 \cdot 5 \cdot 2}{25}} = \frac{3}{5}\sqrt{10}.$

2. $\sqrt{\frac{1}{2}}$. 4. $\sqrt{\frac{4}{27}}$. 6. $\sqrt{\frac{3}{10}}$. 8. $\sqrt[3]{54}$. 10. $\sqrt{\frac{1}{5} - \frac{1}{6}}$.
3. $\sqrt{27}$. 5. $\sqrt{125}$. 7. $\sqrt{3125}$. 9. $\sqrt{\frac{1}{4} + 2}$. 11. $3\sqrt{32}$.

12. $\frac{3}{4}\sqrt{18a^2}$.

13. $\frac{3}{5}\sqrt{162a^3b^4}$.

14. $\sqrt{\frac{1}{12}+\frac{1}{15}}$.

15. $\sqrt{\frac{9}{16}+\frac{16}{25}}$.

16. $\frac{12}{7}\sqrt{1+\frac{1}{12}}$.

17. $\sqrt{\frac{1}{.375}}$.

18. $\sqrt{\frac{3x}{2}}$.

19. $\sqrt{\frac{36a}{x^2}}$.

20. $\sqrt{\frac{3a^3}{4b^5}}$.

21. $a\sqrt{\frac{y^2}{a}}$.

22. $\frac{x^2}{b}\sqrt[4]{\frac{b^5y}{x^7z}}$.

23. $\sqrt{\frac{6y^5}{75}}$.

24. $7\sqrt{\frac{36x^3}{49a^2}}$.

25. $7\sqrt{\frac{3a}{9x^4}}$.

26. $\sqrt{a^3+4a^2b+4ab^2}$.

27. $\sqrt{\frac{x^3+2ax^2+a^2x}{ax+bx}}$.

28. $\sqrt{3x^3+12x^2+12x}$.

29. $\sqrt{\frac{3a^3-12a^2+12a}{12b-12b^2+3b^3}}$.

30. $\sqrt{\frac{2a^3+12a^2+18a}{3b^3-18b^2+27b}}$.

31. $\sqrt{\frac{x^3+x^2y-xy^2-y^3}{36x+36y}}$.

76. Addition and subtraction of radicals. Radicals that are of the same index and have the same expression under the radical sign are **similar**. Only similar radicals can be united into one term by addition and subtraction. We add radical expressions by the following

Rule. *Reduce the radicals to be added to their simplest form. Add the coefficients of similar radicals and prefix this sum as the coefficient of the corresponding radical in the result.*

A rule precisely similar is followed in subtracting radical expressions.

EXERCISES

Perform the indicated additions and subtractions:

1. $\sqrt{50}+\sqrt{32}+\sqrt{18}$.

Solution.

$$\begin{aligned}\sqrt{50}&=\sqrt{25\cdot 2}=5\sqrt{2}\\ \sqrt{32}&=\sqrt{16\cdot 2}=4\sqrt{2}\\ \sqrt{18}&=\sqrt{9\cdot 2}=3\sqrt{2}\\ \hline \text{Sum}\qquad &=12\sqrt{2}\end{aligned}$$

2. $3\sqrt{3}+5\sqrt{3}$.

3. $12\sqrt{6}-3\sqrt{6}$.

4. $a\sqrt{b}+a\sqrt{b}$.

5. $a\sqrt{b}+b\sqrt{b}$.

6. $x-5\sqrt{x}+2\sqrt{\frac{x}{4}}-2\sqrt[3]{x}+\sqrt[3]{8x}+\sqrt{4x}$.

7. $2\sqrt{45} + 3\sqrt{125} - 6\sqrt{20} + 3\sqrt{80} - 2\sqrt{5}$.

8. $7\sqrt{x} + 2\sqrt{y} - 3\sqrt{y} + 2\sqrt{x} - 12\sqrt{x} + 7\sqrt{y}$.

9. $3\sqrt{4a} - 2\sqrt{9a} + 18\sqrt{\frac{2a}{81}} + 3\sqrt{8a} - 7\sqrt{32a}$.

10. $3\sqrt{x+y} + 2\sqrt{4x+4y} - 5\sqrt{9(x+y)} + 4\sqrt{a^2x + a^2y}$.

11. $2\sqrt{x^2a} - 3\sqrt{am^2} + 2\sqrt{a^3} - 5\sqrt{(x+y)^2a} - 7\sqrt{81a}$.

12. $3\sqrt{x} + 2\sqrt{3x} + 7\sqrt{5x} + 7\sqrt{4x} + 2\sqrt{75x} - 7\sqrt{125x}$.

77. Multiplication and division of radicals. For these processes we have the following

Rule. *Follow the usual laws of operation (section 72), using also the assumption of section 74.*

Reduce each term of the result to its simplest form.

The operations of this section are limited to the case where the radicals are of the same index. Radicals of different indices, as $\sqrt{3}$ and $\sqrt[3]{2}$, must first be reduced to the same index (see section 74).

EXERCISES

1. Multiply $\sqrt{3} + \sqrt{2}$ by $\sqrt{3} + \sqrt{8}$.

Solution.

$$\begin{array}{l} \sqrt{3} + \sqrt{2} \\ \underline{\sqrt{3} + \sqrt{8}\qquad\qquad} \\ 3 + \sqrt{6} + \sqrt{24} + \sqrt{16} \\ = 3 + \sqrt{6} + 2\sqrt{6} + 4 \\ = 7 + 3\sqrt{6}. \end{array}$$

2. Divide $\dfrac{x+y}{\sqrt[3]{xy}}$ by $\sqrt[3]{x} + \sqrt[3]{y}$.

Solution.

$$\frac{\dfrac{x+y}{\sqrt[3]{xy}}}{\sqrt[3]{x} + \sqrt[3]{y}} = \frac{(\sqrt[3]{x} + \sqrt[3]{y})(\sqrt[3]{x^2} - \sqrt[3]{xy} + \sqrt[3]{y^2})}{\sqrt[3]{xy}\,(\sqrt[3]{x} + \sqrt[3]{y})}$$

$$= \frac{\sqrt[3]{x^2}}{\sqrt[3]{xy}} - \frac{\sqrt[3]{xy}}{\sqrt[3]{xy}} + \frac{\sqrt[3]{y^2}}{\sqrt[3]{xy}}$$

$$= \sqrt[3]{\frac{x}{y}} - 1 + \sqrt[3]{\frac{y}{x}}.$$

Perform the indicated operations and simplify:

3. $\sqrt{6} \cdot \sqrt{3}$.

4. $\sqrt{27} \cdot \sqrt{3}$.

5. $\sqrt[3]{2} \cdot \sqrt[3]{12}$.

6. $\sqrt{\frac{3}{5}} \div \sqrt{\frac{5}{3}}$.

7. $(x - b\sqrt{y})^2$.

8. $\sqrt{\frac{42}{55}} \cdot \sqrt{\frac{30}{77}}$.

9. $(\sqrt{5} - \sqrt{2})(\sqrt{2} - \sqrt{3})$.

10. $(-2 + \sqrt{5})^2$.

11. $(2\sqrt{3} + 2)(3\sqrt{6} - \sqrt{2})$.

12. $\sqrt{21a} \cdot \sqrt{42a}$.

13. $(a + b + \sqrt{ab})(\sqrt{a} - \sqrt{b})$.

14. $(7 + 2\sqrt{3})(3 - \sqrt{3})$.

15. $\sqrt{\sqrt{x} + \sqrt{y}} \cdot \sqrt{\sqrt{x} - \sqrt{y}}$.

16. $\sqrt{3 - \sqrt{2}} \cdot \sqrt{3 + \sqrt{2}}$.

17. $\sqrt[3]{x + \sqrt{x^2 - 1}} \cdot \sqrt[3]{x - \sqrt{x^2 - 1}}$.

18. $(a^2 + b^2) \div (a\sqrt[3]{b} + b\sqrt[3]{a})$.

19. $\frac{x}{y} \div \sqrt{xy}$.

20. $(3\sqrt{x} + \sqrt{3y})(\sqrt{x} - \sqrt{3y})$.

21. $\sqrt{x} \cdot \sqrt{\frac{2a}{9x}}$.

22. $x \div \sqrt{\frac{x}{y}}$.

23. $\frac{3}{\sqrt{5}} \div \sqrt{\frac{3}{5}}$.

24. $\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2$.

25. $\left(\sqrt{\frac{m+n}{m+p}} - \sqrt{\frac{m+p}{m+n}}\right)^2$.

26. $\sqrt{a^2 - b^2} \cdot \sqrt{\frac{2a - 2b}{a^3 + a^2b}}$.

27. $\frac{x^4 - y^4}{\sqrt[3]{xy}} \div (x\sqrt[3]{x} - y\sqrt[3]{y})$.

28. $\left(\frac{x}{y} - \frac{y}{x}\right) \div \left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\right)$.

29. $(2\sqrt{10} - \sqrt{20} - 3\sqrt{40} + \sqrt{80})\sqrt{5}$.

30. $(2\sqrt{27} - 3\sqrt{75} + 2\sqrt{48} + \sqrt{3})\sqrt{3}$.

31. $(3\sqrt{63} + 7\sqrt{56} + 2\sqrt{7} - 5\sqrt{126})\sqrt{7}$.

32. $(\sqrt{2x + 3} + 2\sqrt{x})(\sqrt{2x + 3} - 2\sqrt{x})$.

33. $[(\sqrt{2} + \sqrt{3} - \sqrt{5})(\sqrt{2} - \sqrt{3} - \sqrt{5})]^2$.

34. $(3\sqrt{42} + 2\sqrt{7} + 3\sqrt{2})(\sqrt{27} + \sqrt{7} - \sqrt{14})$.

35. $(2\sqrt{5} + \sqrt{8} - \sqrt{12})(\frac{1}{3}\sqrt{30} - \frac{2}{3}\sqrt{2} + \sqrt{3})$.

36. $\left(\sqrt{\frac{x+y}{3}}+\sqrt{\frac{x-y}{3}}\right)\left(\sqrt{\frac{x+y}{3}}-\sqrt{\frac{x-y}{3}}\right)$.

37. Find the value of

$$\tfrac{1}{3}\sqrt{75}+2\sqrt{\tfrac{4}{3}}-7\sqrt{15}+2\sqrt{8-\sqrt{2}}\cdot\sqrt{8+\sqrt{2}}$$

to three decimal places.

78. Rationalization. The process of rendering the irrational numerator (or denominator) of a fractional expression rational without altering the value of the fraction is called the **rationalization** of the numerator (or denominator) of the fraction.

This is usually accomplished by multiplying both the numerator and the denominator of the given fraction by a properly chosen radical expression called the **rationalizing factor.**

The principles in accordance with which this rationalizing factor is selected are:

Principle I. Since $(a+b)(a-b)=a^2-b^2$, *the rationalizing factor of* $\sqrt{x}\pm\sqrt{y}$ *is* $\sqrt{x}\mp\sqrt{y}$.

Principle II. Since $(a^2-ab+b^2)(a+b)=a^3+b^3$, *the rationalizing factor of* $\sqrt[3]{x}+\sqrt[3]{y}$ *is* $\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}$, *and conversely.*

Since $(a^2+ab+b^2)(a-b)=a^3-b^3$, *the rationalizing factor of* $\sqrt[3]{x}-\sqrt[3]{y}$ *is* $\sqrt[3]{x^2}+\sqrt[3]{xy}+\sqrt[3]{y^2}$, *and conversely.*

EXERCISES

Rationalize the denominators of the following:

1. $\dfrac{\sqrt{m}+\sqrt{n}}{\sqrt{m}-\sqrt{n}}$.

Solution. By Principle I, the factor which will render the denominator rational is $\sqrt{m}+\sqrt{n}$.

Thus
$$\frac{\sqrt{m}+\sqrt{n}}{\sqrt{m}-\sqrt{n}}=\frac{\sqrt{m}+\sqrt{n}}{\sqrt{m}-\sqrt{n}}\cdot\frac{\sqrt{m}+\sqrt{n}}{\sqrt{m}+\sqrt{n}}$$
$$=\frac{m+2\sqrt{mn}+n}{m-n}.$$

2. $\dfrac{1}{3+\sqrt{3}-\sqrt{5}}$.

Solution. This problem requires a twofold rationalization.

$$\frac{1}{3+\sqrt{3}-\sqrt{5}}=\frac{(3+\sqrt{3})+\sqrt{5}}{[(3+\sqrt{3})-\sqrt{5}][(3+\sqrt{3})+\sqrt{5}]}$$
$$=\frac{3+\sqrt{3}+\sqrt{5}}{(3+\sqrt{3})^2-5}=\frac{3+\sqrt{3}+\sqrt{5}}{9+6\sqrt{3}+3-5}$$
$$=\frac{3+\sqrt{3}+\sqrt{5}}{7+6\sqrt{3}}=\frac{(3+\sqrt{3}+\sqrt{5})(7-6\sqrt{3})}{(7+6\sqrt{3})(7-6\sqrt{3})}$$
$$=\frac{21+7\sqrt{3}+7\sqrt{5}-18-18\sqrt{3}-6\sqrt{15}}{49-36\cdot 3}$$
$$=\frac{3-11\sqrt{3}+7\sqrt{5}-6\sqrt{15}}{-59}.$$

3. $\dfrac{1}{\sqrt[3]{3}-\sqrt[3]{5}}$.

Solution. $$\frac{1}{\sqrt[3]{3}-\sqrt[3]{5}}=\frac{1}{\sqrt[3]{3}-\sqrt[3]{5}}\cdot\frac{\sqrt[3]{3^2}+\sqrt[3]{15}+\sqrt[3]{5^2}}{\sqrt[3]{3^2}+\sqrt[3]{15}+\sqrt[3]{5^2}}$$
$$=\frac{\sqrt[3]{3^2}+\sqrt[3]{15}+\sqrt[3]{5^2}}{3-5}$$
$$=-\tfrac{1}{2}(\sqrt[3]{3^2}+\sqrt[3]{15}+\sqrt[3]{5^2}).$$

4. $\dfrac{1}{3-\sqrt{2}}$.

5. $\dfrac{3+\sqrt{5}}{2-\sqrt{5}}$.

6. $\dfrac{\sqrt{2}-\sqrt{7}}{\sqrt{2}+\sqrt{7}}$.

7. $\dfrac{3}{\sqrt[3]{4}+\sqrt[3]{3}}$.

8. $\dfrac{7}{\sqrt[3]{2}+\sqrt[3]{3}}$.

9. $\dfrac{3\sqrt{\frac{1}{3}}}{\sqrt{3}-4\sqrt{\frac{1}{3}}}$.

10. $\sqrt{\dfrac{a-\sqrt{x}}{a+\sqrt{x}}}$.

11. $\sqrt{\dfrac{x-\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}}}$.

12. $\dfrac{3\sqrt{2}}{\sqrt{3}-\sqrt{2}+\sqrt{6}}$.

13. $\dfrac{2-\sqrt{2}+\sqrt{5}}{\sqrt{5}+2\sqrt{2}+\sqrt{3}}$.

14. $\dfrac{3\sqrt{14}}{\sqrt{7}+\sqrt{2}-2\sqrt{3}}$.

15. $\dfrac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}$.

16. $\dfrac{2}{\sqrt{x-y}+\sqrt{x+y}}$.

17. $\dfrac{\sqrt{6}+\sqrt{5}-\sqrt{3}-\sqrt{2}}{\sqrt{6}-\sqrt{5}+\sqrt{3}-\sqrt{2}}$.

18. $\dfrac{\sqrt{(a+b)(a+2)}+\sqrt{(a-b)(a-2)}}{\sqrt{(a+b)(a+2)}-\sqrt{(a-b)(a-2)}}$.

19. Show that $\frac{2\sqrt{5}+3\sqrt{7}}{\sqrt{7}-\sqrt{5}}=30.29+.$

20. Show that $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}=9.898+.$

21. Show that $\frac{\sqrt{2+\sqrt{6}}-\sqrt{7-\sqrt{6}}}{\sqrt{2+\sqrt{6}}+\sqrt{7-\sqrt{6}}}=.005+.$

79. Solution of equations involving radicals. We prove the important

Theorem. *When an equation in x is multiplied by an expression in x, the resulting equation has, in general, solutions which the first one did not possess.*

Let
$$A=0$$
represent an equation containing x which is satisfied by the values $x=a, b, \cdots, n$. Let B be an expression which vanishes when $x=a_1, b_1, \cdots, n_1$. Then the expression
$$A\cdot B=0$$
is satisfied not only when $x=a, b, \cdots, n,$ but also when $x=a_1, b_1, \cdots, n_1$.

Illustration. The equation $x-2=0$ has $x=2$ for its only solution, while the equation $(x-2)(x-3)=0$ has, in addition, the solution $x=3$.

If in the course of a problem it is necessary to multiply an equation by an expression involving the variable, the solutions of the resulting equations must be substituted in the first one to ascertain if any solutions have been introduced which do not satisfy the original equation. Solutions which have been introduced in the process of solving an equation, but which do not satisfy the original equation, are called **extraneous solutions.**

It may be shown in a similar way that raising the equation in x,
$$A=B,$$
to any power may introduce extraneous solutions.

In some instances all the apparent solutions found are extraneous, in which case the original expression was really not an equation at all, since it did not express equality between two expressions which could be equal for any value of the unknown (see section 52).

For example, $\sqrt{x} + 1 = -1$ leads to $\sqrt{x} = -2$, whence $x = 4$. But, when 4 is substituted for x, we get $\sqrt{4} + 1 = 3 = -1$.

EXERCISES

1. Solve $\sqrt{x + 20} - \sqrt{x - 1} = 3$.

Solution.

$$\begin{aligned} \sqrt{x + 20} &= 3 + \sqrt{x - 1}. \\ x + 20 &= 9 + 6\sqrt{x - 1} + x - 1. \\ 12 &= 6\sqrt{x - 1}. \\ \sqrt{x - 1} &= 2. \\ x - 1 &= 4. \\ x &= 5. \end{aligned}$$

Check. $\sqrt{25} - \sqrt{4} = 3.$

2. Solve $\sqrt{x - 2} - \sqrt{x} = 2$.

Solution.

$$\begin{aligned} \sqrt{x - 2} &= 2 + \sqrt{x}. \\ x - 2 &= 4 + 4\sqrt{x} + x. \\ -6 &= 4\sqrt{x}. \\ 2\sqrt{x} &= -3. \\ 4x &= 9. \\ x &= \tfrac{9}{4}. \end{aligned}$$

Check. $\sqrt{\tfrac{1}{4}} - \sqrt{\tfrac{9}{4}} = \tfrac{1}{2} - \tfrac{3}{2} \neq 2$. (Root $\tfrac{9}{4}$ rejected.)

Solve:

3. $\sqrt{2x} - 1 = 1$.

4. $\sqrt{\tfrac{1}{3}x} + 2 = 5$.

5. $\sqrt{5x + 11} = 6$.

6. $\sqrt{3x + 7} = 10$.

7. $\sqrt{x - 3} = \sqrt{4x - 5}$.

8. $\sqrt{x} + \sqrt{3x} = 2$.

9. $\sqrt[3]{7x + 13} = \sqrt[3]{12x + 3}$.

10. $3\sqrt{x} - 2 = 2\sqrt{x}$.

11. $\sqrt{x + 2} + \sqrt{x + 3} = 2$.

12. $\sqrt{12 + 4\sqrt{x - 2}} = 4$.

13. $2\sqrt{x} - \sqrt{2x} = 2 + \sqrt{2}$.

14. $\sqrt{x + 3} + \sqrt{x - 1} = 2$.

15. $\sqrt{9 - 2\sqrt{3x + 4}} = 1$.

16. $\sqrt[4]{x^2 + 3x - 4} = \sqrt{x + 1}$.

17. $x-\sqrt{ax(1+x)+1-x}=1.$

18. $\sqrt{2(x+1)}+\sqrt{4x-4}=2.$

19. $8+\sqrt{(x+2)(x-4)}=x+6.$

20. $\sqrt{3(x+2)}+\sqrt{3x+25}=19.$

21. $\sqrt{x-2}=\sqrt[4]{x^2-6x+1}.$

22. $1+3\sqrt{2x}=\sqrt{6}+3\sqrt{\frac{1}{3}x}.$

23. $\frac{5}{5+\sqrt{x}}=\frac{1}{7-\sqrt{x}}.$

24. $\frac{2+\sqrt{x}}{2-\sqrt{x}}=5.$

25. $\sqrt{3x-3}=\frac{-3(x-3)}{\sqrt{3x-5}}.$

26. $\sqrt{5x+6}=\frac{5x+2}{\sqrt{5x-1}}.$

27. $\frac{\sqrt{1-x}}{1-\sqrt{1-x}}=\frac{\sqrt{1-x}}{1+\sqrt{1-x}}.$

28. $\sqrt{a-x}+\sqrt{b-x}=\frac{b}{\sqrt{b-x}}.$

29. $\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}.$

30. $3\sqrt{2x+7}+\sqrt{x+3}=12x-1.$

31. $\frac{1+2\sqrt{7x+4}}{1+4\sqrt{7x+4}}=\frac{11}{21}.$

32. $\sqrt{4x+1}+\sqrt{x+2}=\sqrt{9x+7}.$

33. $\sqrt{x+4}-\sqrt{2x-1}=\sqrt{x-5}.$

34. $\sqrt{x+1}+\sqrt{x-2}-\sqrt{x+6}=0.$

35. $(\sqrt{x}+2)(\sqrt{x}+1)=\sqrt{x}\,(\sqrt{x}+4).$

36. $\sqrt{x+2}-\sqrt{x+4}=\sqrt{4x+2}.$

37. $\sqrt{a\sqrt{b}}-\sqrt{b\sqrt{a}}=\frac{a\sqrt{b\sqrt{x}}-b\sqrt{a\sqrt{x}}}{\sqrt{x}}.$

38. $\frac{\sqrt{x}\,(a+\sqrt{x})}{(b-\sqrt{x})\sqrt{x}}=\frac{a+1}{b-1}.$

39. $\frac{4\sqrt{x}-2}{5\sqrt{x}-5}=\frac{2\sqrt{x}+4}{3\sqrt{x}+1}.$

40. $\frac{\sqrt{x}-1}{\sqrt{x}+1}=\frac{\sqrt{2}}{\sqrt{3}}.$

41. $\sqrt{x-2}=\sqrt{x}+2.$

42. $\sqrt{x-1}+\sqrt{x+4}=5.$

43. $\sqrt{3x+1}-\sqrt{3x-1}=1.$

44. $\sqrt{x+12}+\sqrt{x-12}=12.$

45. $\sqrt{3x-2}+\sqrt{3x+5}=5.$

46. $\sqrt{x+2}-\sqrt{x-2}=\sqrt{x+1}+\sqrt{x-1}.$

47. $\sqrt{3x-2}+\sqrt{3x+5}=7.$

48. $\sqrt{5x+4}+\sqrt{5x-1}-\sqrt{5x+20}=0.$

49. $\sqrt{x+1}+\sqrt{x+2}=\sqrt{-x}.$

50. $\sqrt{x-2}+\sqrt{x+2}=0.$

51. $\frac{2-\sqrt{x}}{\sqrt{2x}}=\frac{3\sqrt{x}-2}{\sqrt{2}}.$

CHAPTER VII

THEORY OF INDICES

80. Negative exponents. We have already seen (section 16) that

$$a^m \cdot a^n = a^{m+n} \tag{1}$$

when m and n are positive integers. We now assume that this law still holds when one or both of the numbers m and n are negative or fractional.

If we let $$a^{-m} = \frac{1}{a^m},$$

then $$\frac{a^n}{a^m} = a^n\left(\frac{1}{a^m}\right) = a^n \cdot a^{-m} = a^{n-m},$$

since law (1) holds when m and n are any integers. This notation may be expressed verbally in the following

Principle. *A factor of the numerator or the denominator of a fraction may be changed from the numerator to the denominator, or vice versa, if the sign of its exponent be changed.*

81. Fractional exponents. Since (section 73) $\sqrt{a} \cdot \sqrt{a} = a$, it is natural to devise a notation for $\sqrt{a}$ suggested by law (1).

If we let $$\sqrt{a} = a^{\frac{1}{2}},$$

we have $$\sqrt{a} \cdot \sqrt{a} = a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a.$$

Furthermore, if we let

$$\sqrt[n]{a} = a^{\frac{1}{n}},$$

it would be consistent with law (1) to write

$$\left(a^{\frac{1}{n}}\right)^2 = a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} = a^{\frac{1}{n}+\frac{1}{n}} = a^{\frac{2}{n}}.$$

This notation we shall assume in general. Thus

$$(\sqrt[n]{a})^m = \left(a^{\frac{1}{n}}\right)^m = a^{\frac{m}{n}}.$$

With the adoption of this notation we can attach a meaning to any real number with any rational number for its exponent. This notation may be expressed verbally in the following

Principle. *The numerator of a fractional exponent indicates a power; the denominator, a root.*

82. Résumé of formulas in exponents and radicals.

I. *Definitions.*

1. $(\sqrt[n]{a})^n = a$. Thus $(\sqrt[3]{2})^3 = 2$.
2. $a^{\frac{1}{n}} = \sqrt[n]{a}$. Thus $2^{\frac{1}{3}} = \sqrt[3]{2}$.
3. $\dfrac{1}{a^n} = a^{-n}$. Thus $\dfrac{1}{2^3} = 2^{-3}$.

II. *General laws for any index.*

1. $a^n \cdot a^m = a^{n+m}$. Thus $2^3 \cdot 2^5 = 2^8$.
2. $a^n \cdot b^n = (ab)^n$. Thus $2^3 \cdot 5^3 = (10)^3$.
3. $(a^n)^m = a^{nm}$. Thus $(2^3)^2 = 2^6$.
4. $(a^n)^m = (a^m)^n$. Thus $(2^3)^2 = (2^2)^3 = 2^6$.
5. $(a^n)^{mp} = (a^{nm})^p$. Thus $(2^2)^{3 \cdot 4} = (2^{2 \cdot 3})^4 = 2^{24}$.

III. *General laws in notation of radical and fractional exponents.*

1. $\sqrt[n]{a}\,\sqrt[m]{a} = a^{\frac{1}{n}} \cdot a^{\frac{1}{m}} = a^{\frac{1}{n}+\frac{1}{m}}$. Thus $\sqrt{2} \cdot \sqrt[3]{2} = 2^{\frac{1}{2}+\frac{1}{3}} = 2^{\frac{5}{6}}$.
2. $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$. Thus $\sqrt[3]{2} \cdot \sqrt[3]{5} = \sqrt[3]{10}$.
3. $\sqrt[n]{a^n b} = a\,\sqrt[n]{b}$. Thus $\sqrt[3]{2^3 \cdot 5} = 2\,\sqrt[3]{5}$.
4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$. Thus $\sqrt[4]{\sqrt[3]{2}} = \sqrt[12]{2}$.
5. $(\sqrt[m]{a})^n = \sqrt[m]{a^n}$. Thus $(\sqrt[3]{2})^2 = \sqrt[3]{2^2}$.

83. Principal roots. The ideas of sections 69 and 74, in which the principal root of a number is defined in terms of the radical notation, are also applied to the notation of fractional exponents.

Thus $4^{\frac{1}{2}} = 2$, while $-4^{\frac{1}{2}} = -2$.
Also $(-8)^{\frac{1}{3}} = -2$, while $8^{\frac{1}{3}} = 2$.

As a consequence of this notation, when a root is extracted on both sides of an equation, the principal root must be taken on each side in order that the equation may still be valid. This is the case both for the radical and the fractional notation.

ORAL EXERCISES

Find the numeric value of the following:

1. $\sqrt{4}$.
2. $-\sqrt{9}$.
3. $\sqrt{16}$.
4. $\sqrt[3]{27}$.
5. $\sqrt[3]{-27}$.
6. $\sqrt[4]{81}$.
7. $-\sqrt[4]{625}$.
8. $\sqrt[3]{-8}$.
9. $\sqrt[4]{16}$.
10. $-\sqrt[5]{-32}$.
11. $-\sqrt{64}$.
12. $-\sqrt[3]{-125}$.
13. $9^{\frac{1}{2}}$.
14. $27^{\frac{1}{3}}$.
15. $64^{\frac{1}{2}}$.
16. $8^{\frac{2}{3}}$.
17. $64^{\frac{1}{3}}$.
18. $25^{\frac{3}{2}}$.
19. $81^{\frac{3}{4}}$.
20. $(\frac{1}{9})^{\frac{1}{2}}$.
21. $(\frac{1}{4})^{\frac{1}{2}}$.
22. $(\frac{1}{8})^{\frac{1}{3}}$.
23. $(-\frac{1}{27})^{\frac{1}{3}}$.
24. $(\frac{1}{25})^{\frac{3}{2}}$.

Read as if expressed in radical form:

25. $a^{\frac{1}{2}}$.
26. $x^{\frac{2}{3}}$.
27. $k^{\frac{7}{2}}$.
28. $(ab)^{\frac{2}{3}}$.
29. $2\,x^{\frac{1}{2}}$.
30. $(3\,p)^{\frac{1}{3}}$.
31. $7\,xy^{\frac{2}{3}}$.
32. $8\,ab^{\frac{3}{4}}$.
33. $2\,a^{\frac{1}{2}}b^{\frac{1}{3}}$.
34. $5\,x^{\frac{2}{3}}y^{\frac{1}{2}}$.
35. $m(m-n)^{\frac{1}{3}}$.
36. $2a(x+y)^{\frac{2}{3}}$.
37. $x^{\frac{1}{y}}$.
38. $3^{\frac{1}{a}}b^{\frac{1}{a}}$.
39. $2\,a^{\frac{1}{2}}b^{\frac{2}{a}}$.
40. $11\,x^{\frac{m}{n}}y^{\frac{n}{m}}$.

Read as if expressed with fractional exponents:

41. $\sqrt{x^3}$.
42. $\sqrt[3]{x^2}$.
43. $\sqrt{ab^3}$.
44. $\sqrt[4]{m^3x^5}$.
45. $2\sqrt{a}$.
46. $3\sqrt[3]{7\,x^2}$.
47. $\sqrt[4]{(x+y)^6}$.
48. $\sqrt[4]{(x+y)^2}$.
49. $\sqrt[3]{a^2+2\,a^3b}$.
50. $m\sqrt[3]{a^4(m^2-n^2)}$.

EXERCISES

Express in simplest form with positive exponents:

1. $\dfrac{25\,a^{-2}b^2c^{\frac{5}{2}}}{5\,ab^3c^{-\frac{1}{2}}}$.

Solution. By section 80,

$$\frac{25\,a^{-2}b^2c^{\frac{5}{2}}}{5\,ab^3c^{-\frac{1}{2}}}=\frac{5\cdot c^{\frac{5}{2}}\cdot c^{\frac{1}{2}}}{a\cdot a^2\cdot b^3\cdot b^{-2}}$$

$$=\frac{5\,c^3}{a^3b}.$$

2. $\dfrac{x^2}{x^{-3}}$.

3. $\dfrac{y^{-1}}{y^{-2}}$.

4. $\dfrac{x^{-1}y^{-3}}{x^{-2}y^{-1}}$.

5. $\dfrac{x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{1}{4}}}{\sqrt{xy^3}\cdot\sqrt[4]{z^3}}$.

6. $\dfrac{\sqrt[5]{(a^{15}b^{10})^2}}{(a^{\frac{2}{3}}b^{\frac{4}{3}})^{\frac{3}{2}}}$.

7. $\dfrac{15\,x^{-2}y^{-3}z^{-1}}{25\,x^2y^{-4}z^{-2}}$.

8. $\dfrac{2\,a^{-1}b^{-3}}{3\,a^2b^{-4}}\cdot\dfrac{6\,ab^{-2}}{5\,a^{-2}b}$.

9. $\dfrac{a^{\frac{1}{2}}b^{-\frac{1}{2}}c\sqrt{ab}\cdot\sqrt[3]{a}}{\sqrt[3]{bc}\cdot a^{\frac{1}{3}}b^{\frac{1}{2}}c^2}$.

10. $m^{\frac{3}{4}}n^{\frac{5}{2}}p^{\frac{2}{3}}\sqrt[3]{m^2p^4q^{-1}}$.

11. $\dfrac{2\,a^{-b}\cdot x^{4c}}{3\,y^{-1}\cdot b^2}\cdot\dfrac{5\,y^{-2}\cdot b^{-3}}{3\,a^b\cdot x^{2c}}$.

Arrange in order of magnitude:

12. $\sqrt{\frac{3}{2}},\ \sqrt[3]{\frac{2}{3}},\ \sqrt[4]{\frac{7}{2}}$.

Solution.

$$\sqrt{\frac{3}{2}}=\sqrt[12]{\frac{3^6}{2^6}}=\sqrt[12]{\frac{729}{64}}.$$

$$\sqrt[3]{\frac{2}{3}}=\sqrt[12]{\frac{2^4}{3^4}}=\sqrt[12]{\frac{16}{81}}.$$

$$\sqrt[4]{\frac{7}{2}}=\sqrt[12]{\frac{7^3}{2^3}}=\sqrt[12]{\frac{343}{8}}.$$

The order of magnitude is $\sqrt[4]{\frac{7}{2}},\ \sqrt{\frac{3}{2}},\ \sqrt[3]{\frac{2}{3}}$.

13. $\sqrt[3]{.3},\ \sqrt{.1}$.

14. $\sqrt[4]{6},\ \sqrt[3]{5},\ \sqrt{3}$.

15. $\sqrt{2},\ \sqrt[3]{3},\ \sqrt[4]{5}$.

16. $\sqrt{3},\ \sqrt[6]{10},\ \sqrt[12]{15},\ \sqrt[3]{5}$.

17. $\sqrt{2},\ \sqrt[3]{3},\ \sqrt[5]{5},\ \sqrt{\frac{3}{2}}$.

18. $\sqrt{7},\ \sqrt[3]{9},\ \sqrt[6]{18},\ \sqrt[12]{100}$.

Perform the indicated operations:

19. $\sqrt{2}\cdot\sqrt[3]{3}\cdot\sqrt[6]{5}$.

Solution.

$$\begin{aligned}\sqrt{2}\cdot\sqrt[3]{3}\cdot\sqrt[6]{5}&=2^{\frac{1}{2}}\cdot 3^{\frac{1}{3}}\cdot 5^{\frac{1}{6}}\\&=2^{\frac{3}{6}}\cdot 3^{\frac{2}{6}}\cdot 5^{\frac{1}{6}}\\&=\sqrt[6]{8\cdot 9\cdot 5}\\&=\sqrt[6]{360}.\end{aligned}$$

20. $\sqrt{2}\cdot\sqrt[3]{3}$.

21. $\dfrac{\sqrt{3}\cdot\sqrt[6]{5}}{\sqrt[3]{4}}$.

22. $\dfrac{\sqrt{2}\cdot\sqrt[4]{3}}{\sqrt[3]{3}\cdot\sqrt{5}}$.

23. $\dfrac{\sqrt[3]{5}\cdot\sqrt{6}}{\sqrt{2}\cdot\sqrt[3]{7}}$.

24. $\dfrac{\sqrt[6]{3}\cdot\sqrt{8}}{\sqrt[12]{4}\cdot\sqrt[3]{7}}$.

25. $\dfrac{\sqrt{3}\cdot\sqrt[4]{90}}{\sqrt{45}\cdot\sqrt[8]{625}}$.

84. Operations with radical polynomials. These operations follow the rules for the same operations previously given, provided the assumptions and principles of sections 81–82 are observed.

EXERCISES

Divide:

1. $x^{\frac{2}{5}} - y^{\frac{2}{5}}$ by $x^{\frac{1}{5}} - y^{\frac{1}{5}}$.

2. $x^{\frac{3}{7}} + y^{\frac{3}{7}}$ by $x^{\frac{1}{7}} + y^{\frac{1}{7}}$.

3. $x^{\frac{2}{3}} - y^{\frac{2}{3}}$ by $\sqrt[3]{x} + \sqrt[3]{y}$.

4. $\sqrt{\frac{4}{3}\sqrt{x^3}} - \sqrt{\frac{3}{4}\sqrt{x^3}}$ by $3\sqrt{\frac{3}{x}}$.

Multiply:

5. $2x^{\frac{1}{2}} + 3y^{\frac{1}{2}}$ by $x^{\frac{1}{2}} - 2y^{\frac{1}{2}}$.

6. $\sqrt[3]{x^2} + \sqrt[3]{y^2}$ by $\sqrt[3]{x^2} - \sqrt[3]{y^2}$.

7. $\sqrt[3]{x^2} + 2\sqrt{x^3} + \frac{2}{3}\sqrt{x} + 3\sqrt[3]{x}$ by $\sqrt[3]{x} + \sqrt{x}$.

8. $3x^{\frac{1}{2}}y^{-1} + 2x^{-1}y^{\frac{1}{2}} + 3$ by $x^{-\frac{1}{2}} + 3y^{-\frac{1}{2}}$.

9. $3x^3 - 2x^2y^{\frac{1}{2}} + 3xy + 2y^{\frac{3}{2}}$ by $x^{-3} + 3y^{-\frac{1}{2}}$.

Extract the square root of:

10. $9x + 6x^{\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$.

11. $4x + 49y - 28\sqrt{xy}$.

12. $x^3 + 4\sqrt{x^3y^5} + 4y^5$.

13. $x^2 + 2x^{\frac{3}{2}}y^{\frac{1}{2}} + 3xy + 2x^{\frac{1}{2}}y^{\frac{3}{2}} + y^2$.

14. $x^4 + y^{-2} - x^2y^{-1} + 2xy^{-\frac{3}{2}} - 2x^3y^{-\frac{1}{2}}$.

15. $4x^4 + 28x^3y^{\frac{1}{2}} + 37x^2y - 42xy^{\frac{3}{2}} + 9y^2$.

16. $4x^2 + 12x^{\frac{5}{3}} - 19x^{\frac{4}{3}} - 34x + 61x^{\frac{2}{3}} - 28x^{\frac{1}{3}} + 4$.

17. $\frac{4}{49}x^2y^{-2} - \frac{15}{2}yx^{-1} + \frac{9}{16}y^2x^{-2} - \frac{20}{7}xy^{-1} + 25\frac{3}{7}$.

Simplify:

18. $\dfrac{6x + 3x^{\frac{1}{2}}y^{\frac{1}{2}} - 4x^{\frac{1}{2}}y^{-\frac{1}{2}} - 2}{3x^{\frac{1}{2}} - 2y^{-\frac{1}{2}}}$.

19. $\dfrac{6a^{\frac{3}{2}} + 9a^{\frac{1}{2}}b - 4ab^{\frac{1}{2}} - 6b^{\frac{3}{2}}}{2a + 3b}$.

20. $\dfrac{3x + 3x^{-1} - 6}{x^{\frac{3}{2}} - 3x^{\frac{1}{2}} + 3x^{-\frac{1}{2}} - x^{-\frac{3}{2}}}$.

CHAPTER VIII

LOGARITHMS

85. Introduction. Closely related to the topic of exponents, which was treated in the last chapter, is the subject of logarithms. This theory leads to the most important labor-saving device that has been introduced into mathematics in modern times. Except for the adoption of the Arabic and decimal notation, the invention and use of logarithms probably stands alone both for the elegance of its methods and for the efficiency with which it may be employed in numeric computation.

86. Generalized powers. If b and c are integers, we can easily compute b^c.

For example, $6^4 = 1296$, and $3^5 = 243$.

When c is not an integer but a fraction, we can compute the value of b^c to any desired degree of accuracy. Thus if $b = 2$ and $c = \frac{3}{2}$, we have $2^{\frac{3}{2}} = \sqrt{2^3} = \sqrt{8}$, which we can find to any number of decimal places. If, however, the exponent is an irrational number, as $\sqrt{2}$, we have shown no method of computing the expression. Since, however, $\sqrt{2}$ was seen (section 71) to be the limit approached by the series of numbers

$$1, 1.4, 1.41, 1.414, \cdots,$$

it turns out that $5^{\sqrt{2}}$ is the limit approached by the numbers

$$5^{1.}, 5^{1.4}, 5^{1.41}, 5^{1.414}, \cdots.$$

The computation of such a number as $5^{1.41}$ would be somewhat laborious, but could be performed, since $5^{1.41} = 5^{\frac{141}{100}} = \sqrt[100]{5^{141}}$. Thus, if we let $x = 5^{1.41}$, it appears that x is a root of the equation $x^{100} = 5^{141}$, which could be solved by the methods of Chapter XVII.

In this particular case we see that $5^{\sqrt{2}}$ is the limit approached by a series of numbers in which the exponents are the successive approximations to $\sqrt{2}$ obtained by the process of extracting the square root. In a similar manner we could always express the meaning of b^c, where b is a positive integer and c is any irrational number.

Assumption. We assume that the laws of operation which we have adopted for rational exponents hold when the exponents are irrational.

Thus $b^c \cdot b^d = b^{c+d}$, $\frac{b^c}{b^d} = b^{c-d}$, $(b^d)^c = (b^c)^d = b^{cd}$, where c and d are any numbers, rational or irrational.

87. Logarithms. We have just seen that when b and c are given, a number a exists such that $b^c = a$. We now consider the case where a and b are given and c remains to be found. For example, let $a = 8$ and $b = 2$. If $2^c = 8$, we see immediately that $c = 3$. If $a = 16$ and $b = 2$, then $2^c = 16$, and $c = 4$. If $a = 10$ and $b = 2$, consider the equation $2^c = 10$. If we let $c = 3$, we obtain $2^3 = 8$. If we let $c = 4$, we obtain $2^4 = 16$. If, then, any number c exists such that $2^c = 10$, it must evidently lie between 3 and 4. To prove the existence of such a number is beyond the scope of this chapter, but we make the following

Assumption. There always exists a real number x which satisfies the equation

$$b^x = a, \tag{1}$$

where a and b are positive numbers, provided $b \neq 1$.

Since any real number may be expressed approximately in terms of a decimal fraction, this number x may be so expressed.

Definition. The power to which a given number, called the base, must be raised to equal a second number is called the ***logarithm*** *of the second number.*

In (1) x is the logarithm of a for the base b.

This is abbreviated into

$$x = \log_b a. \tag{2}$$

Since $2^3 = 8$, $10^2 = 100$, $3^{-2} = \frac{1}{9}$, and $4^0 = 1$, we have $\log_2 8 = 3$, $\log_{10} 100 = 2$, $\log_3 \frac{1}{9} = -2$, and $\log_4 1 = 0$.

The number a in (1) or (2) is called the *antilogarithm* of x for the base b.

EXERCISES

In the following name the base, the logarithm, and the antilogarithm, and write in form (2):

1. $10^6 = 1{,}000{,}000$.

Solution. Base = 10, logarithm = 6, antilogarithm = 1,000,000,

$$\log_{10} 1{,}000{,}000 = 6.$$

2. $3^2 = 9$.
3. $2^5 = 32$.
4. $4^4 = 256$.
5. $5^3 = 125$.
6. $7^6 = 117{,}649$.
7. $6^5 = 7776$.

Find the logarithms of the following numbers to the base 2:

8. 16.

Solution. $16 = 2^4$.

$\log_2 16 = 4$.

9. 64.
10. 8.
11. 1.
12. $\frac{1}{2}$.
13. $\frac{1}{32}$.
14. $\frac{1}{128}$.

Find the logarithms of the following numbers to the base 3:

15. 27.

Solution. $27 = 3^3$.

$\log_3 27 = 3$.

16. 243.
17. 729.
18. 1.
19. $\frac{1}{9}$.
20. $\frac{1}{81}$.
21. $\frac{1}{3}$.
22. $\frac{1}{27}$.

What base must be used to make the following statements correct?

23. $\log 9 = 2$.

Solution. $9 = b^2$.

$b = 3$.

$\log_3 9 = 2$.

24. $\log 8 = 3$.
25. $\log 121 = 2$.
26. $\log 36 = 2$.
27. $\log 27 = 3$.
28. $\log 2 = 1$.

Write the following expressions in form (1):

29. $\log_2 4 = 2$.
30. $\log_3 9 = 2$.
31. $\log_5 125 = 3$.
32. $\log_9 81 = 2$.
33. $\log_2 32 = 5$.
34. $\log_7 343 = 3$.
35. $\log_{10} 1000 = 3$.
36. $\log_3 3 = 1$.

88. Operations on logarithms. By means of the law expressed in the assumption, section 87, we prove certain theorems that have made the use of logarithms the most helpful device employed in mathematical computation.

Theorem I. The logarithm of the product of two numbers is the sum of their logarithms.

Let $\log_b a = x$,

and $\log_b c = y$.

Then by (1) and (2), section 87,

$$b^x = a,$$

and $b^y = c$.

Multiplying (by assumption, section 86),

$$b^{x+y} = a \cdot c,$$

or, by (1) and (2), $\log_b a \cdot c = x + y$.

Theorem II. The logarithm of the nth power of a number is n times the logarithm of the number.

Let $\log_b a = x$,

or $b^x = a$.

Raising both sides to the nth power,

$$(b^x)^n = b^{nx} = a^n,$$

or $\log_b a^n = nx$.

Illustration. $\log_{10} 100 = 2$, since $10^2 = 100$,

and $\log_{10} 1000 = 3$, since $10^3 = 1000$.

By Theorem 1, $\log_{10} 100{,}000 = 5$,

which is evidently true, since $10^5 = 100{,}000$.

Theorem III. *The logarithm of the quotient of two numbers is the difference between the logarithms of the numbers.*

Let $\log_b a = x,$

and $\log_b c = y.$

Then $b^x = a,$

and $b^y = c.$

Dividing, $b^{x-y} = \dfrac{a}{c},$

or $\log_b \dfrac{a}{c} = x - y.$

Theorem IV. *The logarithm of the real nth root of a number is the logarithm of the number divided by n.*

Let $\log_b a = x,$

or $b^x = a.$

Extracting the nth root,

$$(b^x)^{\frac{1}{n}} = b^{\frac{x}{n}} = \sqrt[n]{a},$$

or
$$\log_b \sqrt[n]{a} = \frac{x}{n}.$$

EXERCISES

Given $\log_{10} 2 = .301$, $\log_{10} 3 = .477$, $\log_{10} 5 = .699$, find:

1. log* 15.

Solution. By Theorem I,

$$\begin{aligned}\log 15 &= \log 3 + \log 5\\ &= .477 + .699\\ &= 1.176.\end{aligned}$$

2. log 10.

Hint. $10 = 2 \cdot 5$.

3. log 6.

4. log 30.

5. log 20.

6. log $\frac{3}{2}$.

7. log $\sqrt{2}$.

8. log $\sqrt[3]{5}$.

9. log 25.

10. log $(\sqrt[3]{2} \cdot \sqrt[3]{3} \cdot \sqrt{5^3})$.

11. log $(625 \sqrt[4]{24})$.

12. log $(\sqrt[3]{25} \cdot \sqrt{6})$.

13. log 1000.

14. log 1,000,000.

15. log 25,000.

---

* The base 10 is employed unless otherwise specified.

89. Common, or Briggs, system of logarithms. Experience has taught that, for purposes of computation, logarithms of which the base is 10 are much more readily employed than logarithms to any other base. It is common practice to use logarithms having 10 as a base unless special considerations require some other base. We shall therefore assume a base 10 in all our work for the remainder of this chapter. We may write as follows the equations which show the numbers of which a few of the smaller integers are the logarithms for the base 10:

$$\begin{array}{llll}
\text{Since} & 10^5 = 100{,}000 & \text{we have} & \log 100{,}000 = 5. \\
& 10^4 = 10{,}000 & & \log 10{,}000 = 4. \\
& 10^3 = 1000 & & \log 1000 = 3. \\
& 10^2 = 100 & & \log 100 = 2. \\
& 10^1 = 10 & & \log 10 = 1. \\
& 10^0 = 1 & & \log 1 = 0. \\
& 10^{-1} = .1 & & \log .1 = -1. \\
& 10^{-2} = .01 & & \log .01 = -2. \\
& 10^{-3} = .001 & & \log .001 = -3. \\
& \text{etc.} & & \text{etc.}
\end{array}$$

Assuming that as x becomes greater $\log x$ also becomes greater, we see that a number, for example, between 10 and 100 has a logarithm between 1 and 2. In fact, the logarithm of any number not an exact power of 10 consists of a whole-number part and a decimal part.

Thus, since $10^3 < 3421 < 10^4$,
$\log 3421 = 3. + \text{a decimal}.$
Since $10^{-3} < .0023 < 10^{-2}$,
$\log .0023 = -3. + \text{a decimal}.$

Definitions. The whole-number part of the logarithm of a number is called the ***characteristic*** *of the logarithm.*

The decimal part of the logarithm of a number is called the ***mantissa*** *of the logarithm.*

The characteristic of the logarithm of any number may be seen from the table on page 94, from which we immediately deduce the following

Rule. *The characteristic of the logarithm of a number greater than unity is one less than the number of digits to the left of its decimal point.*

Thus the characteristic of the logarithm of 235 is 2, since 235 is between 100 and 1000; the characteristic of the logarithm of 17.31 is 1, since 17.31 is between 10 and 100; and the characteristic of the logarithm of 1928.29 is 3, since 1928.29 is between 1000 and 10,000.

Rule. *The characteristic of the logarithm of a number less than unity is one greater negatively than the number of zeros preceding the first significant figure.*

Thus the characteristic of the logarithm of .03 is -2, that of .0000765 is -5, and that of .372 is -1.

It must be kept constantly in mind that the logarithm of a number less than 1 consists of a negative integer as a characteristic, *plus* a positive decimal fraction as a mantissa. To avoid complication it is desirable to add 10 to and subtract 10 from a logarithm when the characteristic is negative. Thus, for instance, instead of writing the logarithm $-3. + .4672$, we write $10 - 3. + .4672 - 10$, or $7.4672 - 10$. This is convenient when, for example, we wish to divide a logarithm by 2, as by Theorem IV, section 88, we shall wish to do when we extract a square root. Since in the logarithm $-3. + .4672$ the mantissa is positive, it would not be correct to divide -3.4672 by 2, as we should confuse the positive and the negative parts. This confusion is avoided if we use the form $7.4672 - 10$. Hence the result of dividing $7.4672 - 10$ by 2 is $3.7336 - 5$, or $8.7336 - 10$. The actual logarithm which is the result of this division is $-2. + .7336$; that is, the characteristic of the result is -2.

Theorem. *Numbers with the same significant figures which differ only in the position of their decimal points have the same mantissa.*

EXAMPLE

Find the logarithms of 36.51 and 3651.

Solution. Let $10^x = 36.51.$ (1)

Then $x = \log 36.51.$

If we multiply both members of equation (1) by 100, we have

$$10^2 10^x = 10^{2+x} = 3651,$$

or $x + 2 = \log 3651.$

We may therefore say that the logarithm of one of these numbers differs from the logarithm of the other number merely in the characteristic.

In general, numbers with the same significant figures are identical except that one number will be some multiple of 10 times as great as the other. Hence their logarithms differ only by integers, leaving their mantissas the same.

Thus if $\log 47120. = 4.6732$, $\log 47.12 = 1.6732$, and $\log .004712 = 7.6732 - 10$.

ORAL EXERCISES

Supply the missing numbers in the following exercises:

1. $\log 3.76 = .5752$; $\log 37.6 = ?$
2. $\log 12.5 = 1.0969$; $\log 1.25 = ?$
3. $\log 4.2 = .6232$; $\log 42 = ?$
4. $\log .315 = 9.4983 - 10$; $\log 3.15 = ?$
5. $\log 172 = 2.2355$; $\log .172 = ?$
6. $\log 2250 = 3.3522$; $\log 2\frac{1}{4} = ?$
7. $\log 15 = 1.1761$; $\log 1\frac{1}{2} = ?$
8. $\log .00000723 = 4.8591 - 10$; $\log .723 = ?$
9. $\log .00181 = 7.2577 - 10$; $\log 18.1 = ?$
10. $\log 87{,}500{,}000 = 7.9420$; $\log 8750 = ?$
11. $\log 999 = 2.9996$; $\log .999 = ?$
12. $\log .101 = 9.0043 - 10$; $\log 1010 = ?$

EXERCISES

If $\log 2 = .3010$, $\log 3 = .4771$, and $\log 5 = .6990$, find:

1. $\log \sqrt{150}$.

Solution. $\log \sqrt{150} = \log \sqrt{3 \cdot 5 \cdot 10} = \frac{1}{2} \log 3 + \frac{1}{2} \log 5 + \frac{1}{2} \log 10.$

$$\begin{aligned} \tfrac{1}{2} \log 3 &= .2386 \\ \tfrac{1}{2} \log 5 &= .3495 \\ \tfrac{1}{2} \log 10 &= \underline{.5000} \\ \log \sqrt{150} &= 1.0881 \end{aligned}$$

2. $\log .6$.

3. $\log 150$.

4. $\log (15)^3$.

5. $\log (300)^5$.

6. $\log \frac{225}{4}$.

7. $\log \frac{3}{4}$.

8. $\log \left(\frac{6}{5}\right)^2$.

9. $\log \frac{228}{38}$.

10. $\log \sqrt{14.4}$.

90. Use of tables. A table of logarithms contains the mantissas of the logarithms of all numbers having a certain number of significant figures. The table found later in this chapter (pp. 98–99) gives directly the mantissas for all numbers having three significant figures. In the following section a method is given for finding the mantissa corresponding to a number having four significant figures. For this reason the table is known as a "four-place" table. In other words, it is possible to use this table to obtain results correct to four significant figures. The fifth significant figure, however, would not be reliable, and in case we need more accurate results, we should use tables accurate to five or more significant figures. Before each mantissa in the table a decimal point is assumed to stand, but in order to save space it is not printed. To find the logarithm of a number of three or less than three significant figures we apply the following

Rule. Determine the characteristic by the rules of section 89.

Find in column **N** *the first two significant figures of the number. The mantissa required is in the row with these figures.*

Find at the top of the page the last of the three significant figures. The mantissa required is in the column with this figure.

Logarithms of Numbers

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

101777

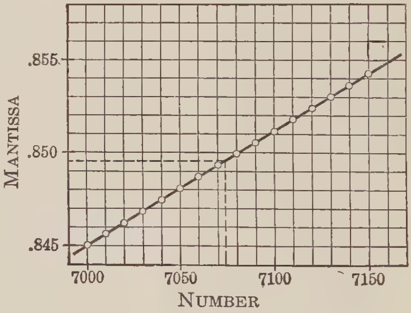
EXERCISES

Find the logarithm of each of the following numbers:

1. 2.
2. 75.
3. 133.
4. 17.4.
5. 21.3.
6. 1.73.
7. .245.
8. .314.
9. 3.14.
10. .784.
11. .0498.
12. .00991.
13. .000432.
14. 1.09.
15. .00000173.
16. 73,400,000.

91. Interpolation. The statement was made in the preceding section that it was possible to read the mantissas of three-place numbers directly from the table on pages 98–99. In case it should become desirable to find the logarithm of a number with four significant figures, we must extend our method of using the table to include this case.

Let us consider the number 7074. This number has four significant figures and lies in value between 7070 and 7080. An inspection of the table will lead us to expect that the mantissa of 7074 will have a value intermediate between the mantissas of 7070 and 7080. The problem is now reduced to that of finding how much greater the mantissa of 7074 is than the mantissa of 7070. To make the matter a little clearer let us construct a graph of numbers from 7000 to 7150 with their corresponding mantissas. Such a curve is shown in the adjacent figure.



It appears from the graph that for the range of values under consideration the curve approximates a straight line. We may therefore assume, without making any appreciable error, that any small part of this curve, as, for instance, the portion between 7070 and 7080, is a straight line.

If this is assumed, we can immediately say that a number halfway between 7070 and 7080 will have a mantissa lying halfway between the mantissas of these two numbers. Similarly, a number lying .4 of the way from 7070 to 7080, or 7074, will have a mantissa .4 of the way from the mantissa of 7070 to the mantissa of 7080.

From the above considerations we may derive the following

Rule. Annex to the proper characteristic the mantissa of the first three significant figures.

Multiply the difference between this mantissa and the next larger mantissa in the table (called the tabular difference) by the remaining figures of the number preceded by a decimal point.

Add this product to the extreme right of the logarithm of the first three figures, rejecting all decimal places beyond the fourth.

In this process of interpolation we have assumed and used the principle that the increase of the logarithm is proportional to the increase of the number. This principle is not strictly true, though for numbers whose first significant figure is greater than 1 the error is so small as not to appear in the fourth place of the logarithm. For numbers whose first significant figure is less than 2 this error may occasionally appear if we find the fourth place by interpolation.

EXERCISES

Find the logarithms of the following:

1. 45.283.

Solution.

$$\begin{array}{lr} \log 45.2 & = 1.6551 \\ & \underline{8} \\ \log 45.283 & = 1.6559 \end{array} \qquad \text{Tabular difference, } \begin{array}{r} 10 \\ \underline{.83} \\ 8.3 \end{array}$$

2. 365.2.

3. 2.397.

4. 32.58.

5. 2084.

6. 53,768.

7. 30,060.

8. .5698.

9. .2543.

10. 8.888.

11. 75.647.

12. .06738.

13. 3.7539.

14. .004569.

15. .0008637.

16. 3.1416.

17. 241.86.

18. 34.385.

19. 90.65.

20. 752.3×87.29.

Solution.

$$\begin{aligned}\log 752.3 &= 2.8764\\ \log 87.29 &= \underline{1.9410}\end{aligned}$$

By Theorem I, section 88,

$$\log (752.3 \times 87.29) = 4.8174$$

21. 75.46×2.365.

22. 2.879×56.756.

23. $10{,}015 \times 236.82$.

24. $.0003795 \times 458.92$.

25. $.5739 \times 25.74$.

26. 1.753×14.375.

27. $\dfrac{.02378}{25.378}$.

Solution.

$$\begin{aligned}\log .02378 &= 8.3762 - 10\\ \log 25.378 &= \underline{1.4044}\\ &\ 6.9718 - 10\end{aligned}$$

28. $\dfrac{245.37}{3.7859}$.

29. $\dfrac{365.28}{.67849}$.

30. $\dfrac{3.7859}{2.0468}$.

31. $\dfrac{.058947}{.038952}$.

32. $\dfrac{20894}{.5692}$.

33. $\dfrac{549.14}{415.76}$.

92. Antilogarithms. We can now find the product or quotient of two numbers (as well as powers and roots of any one number) if we are able to find the number that corresponds to a given logarithm.

For this process we have the following

Rule. *If the mantissa is found exactly in the table, the first two figures of the corresponding number are found in the column* **N** *of the same row, while the third figure of the number is found at the top of the column in which the mantissa is found.*

Place the decimal point so that the rules of section 89 are fulfilled.

EXERCISES

Find the antilogarithms of the following:

1. 2.7752.

Solution. We find the mantissa .7752 in the row corresponding to 59 and in the column headed 6. The significant figures of the antilogarithm are therefore 596. Since the characteristic is 2, we must have three figures to the left of the decimal point. This gives us the antilogarithm of 2.7752 = 596.

2. 1.5502.
3. 3.4082.
4. .4698.
5. 9.6561 – 10.
6. $\bar{2}.8751$.*
7. $\bar{1}.0414$.
8. 7.4843 – 10.
9. 3.5514 – 10.
10. $\bar{3}.9800$.
11. 1.1847.
12. 6.3424.
13. $\bar{5}.3181$.

93. Determination of antilogarithms by interpolation. If the mantissa of a given logarithm lies between two mantissas in the table, we may find the antilogarithm by means of the following

Rule. *Write the number of three figures corresponding to the lesser of the two mantissas between which is the given mantissa.*

Subtract this mantissa from the given mantissa, and divide this number by the tabular difference to one decimal place.

Annex this figure to the three already found, and place the decimal point as the rules in section 89 require.

It should be kept in mind that we may always add any integer to a logarithm, provided we subtract the same number from it. This is useful in two cases:

I. When we wish to subtract a larger logarithm from a smaller one.

II. When we wish to divide a logarithm by an integer which is not contained exactly in the characteristic.

Both these processes are illustrated in the following exercises.

EXERCISES

Find the antilogarithms of the following:

1. 2.7524.

Solution. The mantissa .7524 lies between .7520 and .7528. The tabular difference is therefore 8. The mantissa .7520 corresponds to the antilogarithm 565. Also .7524 – .7520 = 4. Since $\frac{4}{8} = .5$ we attach 5 to the partial antilogarithm 565, and then have 5655. Since the characteristic is 2, the number must have three digits to the left of the decimal point. Therefore the antilogarithm of 2.7524 = 565.5.

2. 2.6604.
3. 4.4864.
4. $\bar{1}.7052$.
5. $\bar{2}.9318$.
6. $\bar{5}.8487$.
7. .3173.

---

* This is another manner of expressing – 2. + .8751.

Perform the following operations by logarithms:

8. $\dfrac{365 \times 50.27}{7.394}$.

Solution.

$$\begin{array}{r@{\;=\;}l}
\log 365 & 2.5623 \\
\log 50.27 & \underline{1.7013} \\
 & 4.2636 \\
\log 7.394 & \underline{.8688} \\
\log \text{result} & 3.3948 \\
\text{Result} & 2482.
\end{array}$$

9. $(3\frac{1}{2})^3$.

10. $(\frac{12}{51})^5$.

11. $(\frac{3869}{2795})^8$.

12. $(5\frac{7840}{8479})^{3.14}$.

13. $2387 \div 5683$.

14. $8570 \div 3487$.

15. $7369 \div .0004672$.

16. $2479 \div .003466$.

17. $\dfrac{6.385 \times 234.7}{538.3}$.

18. $\dfrac{784.5 \times .03785}{35.78}$.

19. $\dfrac{2784 \times .03278}{1.378}$.

20. $\sqrt[3]{\dfrac{28 \times \sqrt{8753}}{1{,}275{,}000}}$.

Solution.

$$\begin{array}{r@{\;=\;}l}
\log 28 & 1.4472 \\
\frac{1}{2}\log 8753 & \underline{1.9711} \\
 & 13.4183 - 10 \\
\log 1{,}275{,}000 & \underline{6.1055} \\
 & 3\overline{)27.3128 - 30} \\
\log \text{result} & 9.1043 - 10 \\
\text{Result} & .1271
\end{array}$$

Since in the subtraction of this problem we must subtract 6 from 3, we have added and subtracted 10 in the minuend (3.4183) to avoid a negative logarithm. Since in the division by 3 we should not be able to divide -10 by 3 exactly, we have added and subtracted 20, so that 3 may be contained in 30 exactly.

21. $\sqrt[4]{3}$.

22. $\sqrt{.05}$.

23. $\sqrt[5]{(.2784)^4}$.

24. $\sqrt[13]{26}$.

25. $(.37)^{\frac{3}{4}}$.

26. $(4.468)^{\frac{1}{4}}$.

27. $\frac{2573}{5639}\sqrt[4]{376}$.

28. $\sqrt[3]{.0046}$.

29. The area of a circle is expressed by the formula $A = \pi r^2$, where A = area, r = radius, and $\pi = 3.1416$. Find by logarithms the area of a circle having a radius of 4.763 inches.

30. Using the formula of problem 29, determine the radius of a circle whose area is 784.9 square feet.

31. The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$, where $V =$ volume, $r =$ radius, and $\pi = 3.1416$. Find the volume of a sphere having a radius of 75.48 feet.

32. Using the formula of problem 31, determine the radius of a sphere having a volume of 127.5 cubic inches.

33. A body falling under the influence of gravity travels a distance expressed by the formula $S = \frac{1}{2} gt^2$, where $S =$ distance traveled in feet, $g =$ the constant acceleration of gravity (32 feet per second per second), and $t =$ the elapsed time in seconds. By the use of logarithms find the length of time it will take a stone to fall 100 feet; 500 feet; 1 mile.

34. The velocity of a freely falling body is expressed by the formula $V = gt$, where $V =$ the velocity in feet per second, $g = 32$ feet per second per second, and $t =$ the elapsed time in seconds. By the use of this formula and the formula in problem 33, determine the speed of a body which has fallen for a distance of 1000 feet; 20 feet; 1 mile.

35. The air resistance of an automobile may be expressed by the approximate formula

$$R = .0042\,V^2A,$$

where $R =$ the resistance of the automobile in pounds, $V =$ the speed of the car in miles per hour, and $A =$ a constant depending on the size of the automobile, which has a usual value of about 25. Using the above formula, compute the resistance of an automobile traveling 30 miles per hour. What is the speed of an automobile whose air resistance is 100 pounds? 150 pounds?

36. The power required to drive a car against the resistance of the air is expressed by the approximate formula

$$HP = .0000112\,V^3A,$$

in which $HP =$ the power in terms of horse power, $V =$ the velocity in miles per hour, and $A =$ a constant with a value under usual conditions of 25. What is the power required to drive a car 60 miles per hour? 40 miles per hour? How fast would a 30 horsepower car be able to travel?

37. The velocity of a jet of water flowing from a round hole in a tank is expressed by the formula

$$V = \sqrt{2\,gh},$$

where V is the velocity of the jet in feet per second, g is the acceleration of gravity, 32.2, and h is the height in feet of the water level in the tank above the hole. Find the value of V when $h = 10$ feet. Find the value of h when $V = 15$ feet per second.

94. Exponential equations. Equations in which the variable occurs only in the exponents may often be solved by the use of logarithms, provided one keeps in mind the fact that

$$\log a^x = x \log a.$$

EXERCISES

Solve the following:

1. $10^{2x+3} = 5$.

Solution. Taking the logarithm of both sides of the equation, we have

$$(2x + 3) \log 10 = \log 5,$$

or, since $\log 10 = 1$, $x = \frac{\log 5 - 3}{2} = \frac{.6990 - 3.}{2} = -1.1505$.

2. $5^x = 3$.

3. $7^y = 24$.

4. $2^{x-1} = 13$.

5. $6^{2x-3} = 9$.

6. $3^{x-1} = 4^{x+1}$.

7. $12^{3x+1} = 7^{x-1}$.

8. $3^x \cdot 4^y = 7$,
$x + y = 8$.

9. $5^y \cdot 4^x = 33$,
$x - y = 15$.

10. $12^{3x-2} \cdot 12^{2y-3} = 12^9$,
$3x + 2y = 10$.

11. $3^{x-1} \cdot 4^{y+1} = 2$,
$5^{x+1} \cdot 2^{y-1} = 6$.

12. $\sqrt[3]{5^{7x-1}} \cdot \sqrt{5^{2y+3}} = 25$,
$\sqrt{7^{2x-3}} \cdot \sqrt[3]{7^{7y+1}} = 13$.

95. Compound interest. If \$2000 is invested at an interest rate of 4% per annum, the interest for one year will be \$2000(.04) = \$80. The total sum invested at the end of one year would be \$2080.

Let P represent, in general, the sum of money in dollars.

Let r represent the rate of annual interest.

Then $P \cdot r =$ the yearly interest on P, and

$$P + Pr = P(1 + r)$$

represents the total investment, principal and interest, at the end of one year.

Similarly, we find that $P(1+r)r$ is the interest for the second year, and hence

$$P(1+r)r+P(1+r)=P(1+r)^2$$

is the total investment at the end of two years.

In general, we may say that

$$A=P(1+r)^n \tag{1}$$

is the total accumulation at the end of n years. If we know r, P, and n, we can find A by the use of (1). If we take the logarithm of both sides of the equation, we have

$$\log A=\log P+n\log(1+r),$$

or

$$n=\frac{\log A-\log P}{\log(1+r)}. \tag{2}$$

Hence, if we know A, P, and r, we can find n.

If the interest is compounded semiannually, we have as interest at the end of the first half-year $P\cdot\frac{r}{2}$, while the entire sum would be $P\left(\frac{r}{2}+1\right)$. Reasoning as above, we find that if the interest is compounded semiannually, the accumulation at the end of n years is

$$A=P\left(\frac{r}{2}+1\right)^{2n}. \tag{3}$$

Similarly,

$$n=\frac{\log A-\log P}{2\log\left(\frac{r}{2}+1\right)}. \tag{4}$$

If the interest is compounded k times a year, we have at the end of n years

$$A=P\left(\frac{r}{k}+1\right)^{kn}, \tag{5}$$

or

$$n=\frac{\log A-\log P}{k\log\left(\frac{r}{k}+1\right)}. \tag{6}$$

PROBLEMS

In such problems as the following, four-place tables are not sufficiently exact to give perfect accuracy. In general, the greater the power to which the expression is raised, the greater will be the inaccuracy in the result.

1. If $1500 is invested at $4\frac{1}{4}$ % interest compounded quarterly for a period of 10 years, what will be the amount by the end of that time?

Solution.

$$A = P\left(\frac{r}{4} + 1\right)^{4n}.$$

$$P = 1500,\ r = .0425,\ n = 10.$$

Hence

$$A = 1500(1.010625)^{40}.$$

$$\begin{aligned} \log 1500 &= 3.1761 \\ 40 \log 1.010625 &= \underline{.1840} \\ \log A &= 3.3601 \\ A &= \$2292. \end{aligned}$$

2. How long before $1000 will have become $2500 at 3 % interest compounded semiannually?

Solution.

$$n = \frac{\log A - \log P}{2 \log \left(\frac{r}{2} + 1\right)}.$$

$$A = 2500,\ P = 1000,\ r = .03.$$

$$n = \frac{3.3979 - 3.}{2(.0065)} = \frac{.3979}{.0130} = 30.607 \text{ years.}$$

$$.607 \text{ year} = 7.284 \text{ months.}$$

$$.284 \text{ month} = 8.52 \text{ days.}$$

$$n = 30 \text{ years, } 7 \text{ months, } 8.52 \text{ days.}$$

3. What will $3000 have amounted to in 5 years at 7 % interest compounded annually?

4. Find the amount of $1 at 4 % interest compounded annually for a period of 100 years.

5. Find the amount of $500 at 6 % interest for 10 years, compounded (*a*) annually; (*b*) semiannually; (*c*) quarterly.

6. How many years will be required for $2000 to double itself at 5 % interest compounded (*a*) annually? (*b*) semiannually? (*c*) biennially?

7. What sum of money invested at 3 % for a period of 25 years will amount to $5000 if interest is compounded annually?

8. What sum of money at 4 % compounded annually will amount to as much as \$25,300 at 6 % compounded semiannually, when both sums are invested for a period of 32 years?

9. If a certain sum of money at $4\frac{1}{2}$ % compounded annually amounts to \$135 less than the same sum at 5 % compounded annually, and both sums were invested for a period of 12 years, find the principal.

10. Which amounts to more, a certain sum at 15 % compounded annually for 5 years, or the same sum at 5 % compounded annually for 15 years? If the principal is \$100, what is the difference?

11. Two sums of money, totaling \$58,125, are invested at $4\frac{3}{4}$ % compounded semiannually for a period of 15 years. The difference in the amounts at the end of that time is \$15,600. Find the original principals.

12. At what rate of interest must money be invested in order to quadruple itself in 15 years if interest is compounded quarterly?

13. At what interest rate must money be invested so that it will double itself in 20 years if interest is compounded annually?

14. How long must \$50,000 be invested at $4\frac{1}{2}$ % compounded monthly to amount to \$60,000?

15. In 1626 the Dutch bought the island of Manhattan for goods valued at \$24. If they had put the money in the savings bank at 4 % interest compounded annually, how much would the money have amounted to by 1926?

16. Two amounts of \$10 each are invested at 4 % for a period of one year. If one sum is compounded weekly and the other sum is compounded monthly, what is the difference in the total amounts at the end of the year?

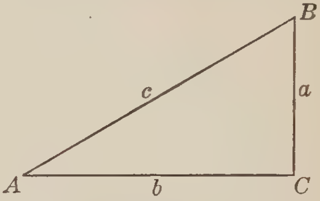
17. In how many years will a certain sum of money amount to ten times itself if left at 5 % interest compounded annually?

18. How long before \$5000 at 4 % compounded annually will amount to as much as \$6000 at 3 % compounded annually, if both amounts are invested at the same time?

19. An investment of \$4000 draws $5\frac{1}{2}$ % interest compounded quarterly until the amount has been doubled. Thereafter the interest is at the rate of $5\frac{3}{4}$ % compounded annually. What will the amount be at the end of 20 years?

20. How long will it take to double a sum of money at 4% interest compounded (*a*) annually? (*b*) semiannually? (*c*) quarterly? (*d*) monthly? (*e*) weekly? Could this money be made to double itself in 15 years if the interest were compounded with sufficient frequency?

96. Trigonometry. In surveying, astronomy, and in countless other applications of science it is necessary to solve triangles. A triangle has six parts, three sides and three angles. When three of these parts, of which one at least must be a side, have been given or have been measured, the other three can be computed by the methods of trigonometry. In developing the principles of trigonometry it is simplest to begin with the right triangle.



In order to attack the problem stated above it is necessary to denote the ratios of the sides of a right triangle by certain names.

Let ABC be any right triangle with the right angle at C.

The sine of the angle $A = \frac{a}{c} = \frac{\text{side opposite}}{\text{hypotenuse}}$.

The cosine of the angle $A = \frac{b}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}}$.

The tangent of the angle $A = \frac{a}{b} = \frac{\text{side opposite}}{\text{side adjacent}}$.

The cotangent of the angle $A = \frac{b}{a} = \frac{\text{side adjacent}}{\text{side opposite}}$.

These definitions are abbreviated as follows:

$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}, \quad \tan A = \frac{a}{b}, \quad \cot A = \frac{b}{a}.$$

The solution of right triangles in which the lengths of the sides are numbers consisting of only two digits is usually performed with the aid of tables of natural sines, tangents, etc.

When the sides are more complicated, tables obtained by taking the logarithms of the natural trigonometric functions are used, as will be described in the following section.

EXERCISES

By means of the table of logarithms on pages 98–99 find the logarithmic functions of the following, and verify by reference to the table on pages 112–113:

1. sin 34° = .5592.

Solution. log .5592 = 9.7476 − 10.
From the table, log sin 34° = 9.7476 − 10.

2. cos 23° = .9205.
3. sin 15° = .2588.
4. tan 69° = 2.6051.
5. cos 46° = .6947.
6. cot 82° = .1405.
7. tan 53° = 1.3270.
8. tan 29° = .5543.
9. sin 75° = .9659.
10. cot 36° = 1.3764.
11. cos 62° = .4695.
12. cot 87° = .0524.
13. cos 38° = .7880.

97. Logarithms of the trigonometric functions. Since the solution of triangles and other uses of trigonometry require both multiplication and division, the actual labor involved is much shortened by using a table of the logarithms of the functions instead of the functions themselves. Interpolation with such a table is precisely the same as with a table of natural functions. In dealing with angles greater than 45° the angles and functions listed at the right and bottom of the table are used.

In using the table on pages 112–113 it is important to note that the sines and cosines of all angles from 0° to 90° are less than one except cosine 0° and sine 90°; also that the tangents of all angles less than 45° and the cotangents of all angles greater than 45° are less than one. Hence the logarithms of these functions will be negative, and minus 10 is understood after each in the table. In order to save space this − 10 is not included in the table.

LOGARITHMS OF TRIGONOMETRIC RATIOS

°	SIN	COS	TAN	COT		°	SIN	COS	TAN	COT	
0°	− ∞	10.0000	− ∞	∞	90°	11°	9.2806	9.9919	9.2887	0.7113	79°
15′	7.6398	.0000	7.6398	2.3602	45′	15′	.2902	.9916	.2987	.7013	45′
30′	.9408	.0000	.9409	.0591	30′	30′	.2997	.9912	.3085	.6915	30′
45′	8.1169	.0000	8.1170	1.8830	15′	45′	.3089	.9908	.3181	.6819	15′
1°	8.2419	9.9999	8.2419	1.7581	89°	12°	9.3179	9.9904	9.3275	0.6725	78°
15′	.3388	.9999	.3389	.6611	45′	15′	.3267	.9900	.3367	.6633	45′
30′	.4179	.9999	.4181	.5819	30′	30′	.3353	.9896	.3458	.6542	30′
45′	.4848	.9998	.4851	.5149	15′	45′	.3438	.9892	.3546	.6454	15′
2°	8.5428	9.9997	8.5431	1.4569	88°	13°	9.3521	9.9887	9.3634	0.6366	77°
15′	.5939	.9997	.5943	.4057	45′	15′	.3602	.9883	.3719	.6281	45′
30′	.6397	.9996	.6401	.3599	30′	30′	.3682	.9878	.3804	.6196	30′
45′	.6810	.9995	.6815	.3185	15′	45′	.3760	.9874	.3886	.6114	15′
3°	8.7188	9.9994	8.7194	1.2806	87°	14°	9.3837	9.9869	9.3968	0.6032	76°
15′	.7535	.9993	.7542	.2458	45′	15′	.3912	.9864	.4048	.5952	45′
30′	.7857	.9992	.7865	.2135	30′	30′	.3986	.9859	.4127	.5873	30′
45′	.8156	.9991	.8165	.1835	15′	45′	.4059	.9855	.4204	.5796	15′
4°	8.8436	9.9989	8.8446	1.1554	86°	15°	9.4130	9.9849	9.4281	0.5719	75°
15′	.8699	.9988	.8711	.1289	45′	15′	.4200	.9844	.4356	.5644	45′
30′	.8946	.9987	.8960	.1040	30′	30′	.4269	.9839	.4430	.5570	30′
45′	.9181	.9985	.9196	.0804	15′	45′	.4337	.9834	.4503	.5497	15′
5°	8.9403	9.9983	8.9420	1.0580	85°	16°	9.4403	9.9828	9.4575	0.5425	74°
15′	.9614	.9982	.9633	.0367	45′	15′	.4469	.9823	.4646	.5354	45′
30′	.9816	.9980	.9836	.0164	30′	30′	.4533	.9817	.4716	.5284	30′
45′	9.0008	.9978	9.0030	0.9970	15′	45′	.4597	.9812	.4785	.5215	15′
6°	9.0192	9.9976	9.0216	0.9784	84°	17°	9.4659	9.9806	9.4853	0.5147	73°
15′	.0369	.9974	.0395	.9605	45′	15′	.4721	.9800	.4921	.5079	45′
30′	.0539	.9972	.0567	.9433	30′	30′	.4781	.9794	.4987	.5013	30′
45′	.0702	.9970	.0732	.9268	15′	45′	.4841	.9788	.5053	.4947	15′
7°	9.0859	9.9968	9.0891	0.9109	83°	18°	9.4900	9.9782	9.5118	0.4882	72°
15′	.1011	.9965	.1045	.8955	45′	15′	.4958	.9776	.5182	.4818	45′
30′	.1157	.9963	.1194	.8806	30′	30′	.5015	.9770	.5245	.4755	30′
45′	.1299	.9960	.1338	.8662	15′	45′	.5071	.9763	.5308	.4692	15′
8°	9.1436	9.9958	9.1478	0.8522	82°	19°	9.5126	9.9757	9.5370	0.4630	71°
15′	.1568	.9955	.1614	.8387	45′	15′	.5181	.9750	.5431	.4569	45′
30′	.1697	.9952	.1745	.8255	30′	30′	.5235	.9743	.5491	.4509	30′
45′	.1822	.9949	.1873	.8127	15′	45′	.5288	.9737	.5551	.4449	15′
9°	9.1943	9.9946	9.1997	0.8003	81°	20°	9.5341	9.9730	9.5611	0.4389	70°
15′	.2061	.9943	.2118	.7882	45′	15′	.5392	.9723	.5669	.4331	45′
30′	.2176	.9940	.2236	.7764	30′	30′	.5443	.9716	.5727	.4273	30′
45′	.2288	.9937	.2351	.7649	15′	45′	.5494	.9709	.5785	.4215	15′
10°	9.2397	9.9934	9.2463	0.7537	80°	21°	9.5543	9.9702	9.5842	0.4158	69°
15′	.2503	.9930	.2573	.7427	45′	15′	.5592	.9694	.5898	.4102	45′
30′	.2606	.9927	.2680	.7320	30′	30′	.5641	.9687	.5954	.4046	30′
45′	.2707	.9923	.2784	.7216	15′	45′	.5689	.9679	.6009	.3991	15′
	COS	SIN	COT	TAN	°		COS	SIN	COT	TAN	°

Logarithms of Trigonometric Ratios

°	SIN	COS	TAN	COT		°	SIN	COS	TAN	COT	
22°	9.5736	9.9672	9.6064	0.3936	68°	34°	9.7476	9.9186	9.8290	0.1710	56°
15′	.5782	.9664	.6118	.3882	45′	15′	.7504	.9173	.8331	.1669	45′
30′	.5828	.9656	.6172	.3828	30′	30′	.7531	.9160	.8371	.1629	30′
45′	.5874	.9648	.6226	.3774	15′	45′	.7559	.9147	.8412	.1588	15′
23°	9.5919	9.9640	9.6279	0.3721	67°	35°	9.7586	9.9134	9.8452	0.1548	55°
15′	.5963	.9632	.6331	.3669	45′	15′	.7613	.9120	.8493	.1507	45′
30′	.6007	.9624	.6383	.3617	30′	30′	.7640	.9107	.8533	.1467	30′
45′	.6050	.9616	.6435	.3565	15′	45′	.7666	.9093	.8573	.1427	15′
24°	9.6093	9.9607	9.6486	0.3514	66°	36°	9.7692	9.9080	9.8613	0.1387	54°
15′	.6135	.9599	.6537	.3463	45′	15′	.7718	.9066	.8652	.1348	45′
30′	.6177	.9590	.6587	.3413	30′	30′	.7744	.9052	.8692	.1308	30′
45′	.6219	.9582	.6637	.3363	15′	45′	.7769	.9038	.8732	.1268	15′
25°	9.6259	9.9573	9.6687	0.3313	65°	37°	9.7795	9.9023	9.8771	0.1229	53°
15′	.6300	.9564	.6736	.3264	45′	15′	.7820	.9009	.8811	.1189	45′
30′	.6340	.9555	.6785	.3215	30′	30′	.7844	.8995	.8850	.1150	30′
45′	.6379	.9546	.6834	.3166	15′	45′	.7869	.8980	.8889	.1111	15′
26°	9.6418	9.9537	9.6882	0.3118	64°	38°	9.7893	9.8965	9.8928	0.1072	52°
15′	.6457	.9527	.6930	.3070	45′	15′	.7918	.8950	.8967	.1033	45′
30′	.6495	.9518	.6977	.3023	30′	30′	.7941	.8935	.9006	.0994	30′
45′	.6533	.9508	.7025	.2975	15′	45′	.7965	.8920	.9045	.0955	15′
27°	9.6570	9.9499	9.7072	0.2928	63°	39°	9.7989	9.8905	9.9084	0.0916	51°
15′	.6607	.9489	.7118	.2882	45′	15′	.8012	.8890	.9122	.0878	45′
30′	.6644	.9479	.7165	.2835	30′	30′	.8035	.8874	.9161	.0839	30′
45′	.6680	.9469	.7211	.2789	15′	45′	.8058	.8858	.9200	.0800	15′
28°	9.6716	9.9459	9.7257	0.2743	62°	40°	9.8081	9.8843	9.9238	0.0762	50°
15′	.6752	.9449	.7302	.2698	45′	15′	.8103	.8827	.9277	.0723	45′
30′	.6787	.9439	.7348	.2652	30′	30′	.8125	.8810	.9315	.0685	30′
45′	.6821	.9429	.7393	.2607	15′	45′	.8148	.8794	.9353	.0647	15′
29°	9.6856	9.9418	9.7438	0.2562	61°	41°	9.8169	9.8778	9.9392	0.0608	49°
15′	.6890	.9408	.7482	.2518	45′	15′	.8191	.8761	.9430	.0570	45′
30′	.6923	.9397	.7526	.2474	30′	30′	.8213	.8745	.9468	.0532	30′
45′	.6957	.9386	.7571	.2429	15′	45′	.8234	.8728	.9506	.0494	15′
30°	9.6990	9.9375	9.7614	0.2386	60°	42°	9.8255	9.8711	9.9544	0.0456	48°
15′	.7022	.9364	.7658	.2342	45′	15′	.8276	.8694	.9582	.0418	45′
30′	.7055	.9353	.7701	.2299	30′	30′	.8297	.8676	.9621	.0379	30′
45′	.7087	.9342	.7745	.2255	15′	45′	.8317	.8659	.9659	.0341	15′
31°	9.7118	9.9331	9.7788	0.2212	59°	43°	9.8338	9.8641	9.9697	0.0303	47°
15′	.7150	.9319	.7831	.2169	45′	15′	.8358	.8624	.9735	.0265	45′
30′	.7181	.9308	.7873	.2127	30′	30′	.8378	.8606	.9772	.0228	30′
45′	.7212	.9296	.7916	.2084	15′	45′	.8398	.8588	.9810	.0190	15′
32°	9.7242	9.9284	9.7958	0.2042	58°	44°	9.8418	9.8569	9.9848	0.0152	46°
15′	.7272	.9272	.8000	.2000	45′	15′	.8437	.8551	.9886	.0114	45′
30′	.7302	.9260	.8042	.1958	30′	30′	.8457	.8532	.9924	.0076	30′
45′	.7332	.9248	.8084	.1916	15′	45′	.8476	.8514	.9962	.0038	15′
33°	9.7361	9.9236	9.8125	0.1875	57°	45°	9.8495	9.8495	0.0000	0.0000	45°
15′	.7390	.9224	.8167	.1833	45′						
30′	.7419	.9211	.8208	.1792	30′						
45′	.7447	.9198	.8249	.1751	15′						
	COS	SIN	COT	TAN	°		COS	SIN	COT	TAN	°

On the other hand, the tangents of angles greater than 45° and the cotangents of angles less than 45° are greater than one, and hence their logarithms are *positive* and are printed in full.

The foregoing can be reduced to one simple statement:

Minus 10 is understood after all logarithms in the table except those in the column at the top of which is the abbreviation COT.

EXERCISES

Compute the missing parts in the following right triangles:

1. $a = 15$, $B = 35°$.

Solution. $\tan B = \frac{b}{a}$, or $b = a \tan B$.

$$\begin{aligned} \log b &= \log a + \log \tan B \\ &= 1.1761 + 9.8452 - 10 \\ &= 11.0213 - 10 \\ &= 1.0213. \\ b &= 10.50. \end{aligned}$$

$$\begin{aligned} \log c &= \log a - \log \sin A \\ &= 1.1761 - 9.9134 + 10 \\ &= 1.2627. \\ c &= 18.31. \\ A &= 55°. \end{aligned}$$

2. $a = 2$, $B = 4°$.

3. $A = 40°$, $c = 12$.

4. $B = 46°$, $b = 5$.

5. $A = 18°$, $b = 13.5$.

6. $B = 79°$, $c = 18.2$.

7. $B = 54°$, $a = 21.3$.

8. $a = 103.6$, $A = 84°$.

9. $A = 34° 15'$, $b = 45$.

10. $B = 40° 38'$, $c = 23$.

11. $A = 39° 52'$, $a = 9.32$.

12. $B = 12° 5'$, $a = 7.35$.

13. $B = 2° 1'$, $c = 108.73$.

14. The line of sight from an observer to the top of a mountain 4 miles distant makes an angle with the horizontal of 3° 17′. How high is the mountain-top above the observer?

15. How many degrees does a stick a foot long subtend when it is at a distance of 1000 feet? Express the result in degrees and minutes.

16. A cloud is known to be at a distance of 5 miles from an observer. If the angle between the line of sight to the cloud and the horizontal is 38° 46′, what is the altitude of the cloud?

17. How long a shadow does a building 525 feet high cast when the sun is 43° 10′ above the horizon?

CHAPTER IX

QUADRATIC EQUATIONS

98. Definition. The general form of the quadratic equation in one variable is

$$ax^2 + bx + c = 0, \tag{1}$$

where a, b, and c represent rational numbers, and where $a \neq 0$. Every quadratic equation in x can be brought to this form by transposing and simplifying.

Thus when $b = 0$ the equation reduces to the form $ax^2 + c = 0$. This is sometimes called a *pure quadratic.*

99. Solution of quadratic equations by factoring. The solution of a quadratic equation consists in finding its roots, that is, the numbers (or expressions involving the coefficients in case the coefficients are literal) which satisfy the equation.

When the left-hand member of an equation can readily be factored, this is the most convenient method of solution. It also illustrates very clearly the meaning and property of the roots of the equation.

EXAMPLE

Solve $x^2 + 5x - 14 = 0$.

Solution. Factoring the left-hand member,

$$(x + 7)(x - 2) = 0.$$

The object in solving an equation is to find numbers which, when substituted for the variable, satisfy the equation. Since zero multiplied by any number is zero (p. 4), any value of x which causes *one* factor of an expression to vanish makes the whole expression vanish. If in this case $x = 2$, the factored form of the equation becomes

$$(2 + 7)(2 - 2) = (2 + 7)0 = 0,$$

and is satisfied. If we let $x = -7$, the other factor becomes zero, and the equation reduces to the identity

$$(-7 + 7)(-7 - 2) = 0(-7 - 2) = 0.$$

Thus the numbers 2 and -7 are solutions of the equation.

The foregoing process suggests the following

Rule. *Transpose all the terms to the left-hand member of the equation.*

Factor that member into linear factors.

The values of the variable that make the factors vanish are the roots of the equation.

EXERCISES

Solve and check the following:

1. $x^2 + 5x + 6 = 0$.
2. $x^2 + x - 6 = 0$.
3. $x^2 + 2x + 1 = 0$.
4. $6x^2 - x = 12$.
5. $6x^2 + 17x = 14$.
6. $6x^2 + 17x = 3$.
7. $4x^2 + 68x = 111$.
8. $7x^2 - 33x = 10$.
9. $3x^2 + 4x + 1 = 0$.
10. $9x^2 - 17x - 2 = 0$.
11. $x^2 - ax = 2a^2$.
12. $x^2 - 5ax + 6a^2 = 0$.
13. $a^2x^2 - 4abx = 5b^2$.
14. $(x + 2)^2 - (x - 3)^2 = 0$.
15. $(x + 5)^2 - (x + 2)^2 = 0$.
16. $(x + a)^2 - (x + 2a)^2 = 0$.
17. $(3x - 2)(x + 1) - (x - 1)(2x - 3) = 2$.
18. $(x + 1)(x - 1) - (2x + 3)(7x - 15) = -236$.
19. $(x - a)(x + 2a) - (x - 2a)(x + a) = 2a$.
20. $(x + 2a)(x - 2a) + (2x - a)(2x + a) = 20 - 5a^2$.

100. Solution of quadratic equations by completing the square. The standard method of solving a quadratic equation consists in bringing the member of the equation which involves the variable into the form of a perfect square, that is, into the form

$$x^2 + 2Ax + A^2.$$

EXAMPLE

Solve $x^2 + 2x - 15 = 0$.

Solution. Transposing 15, $x^2 + 2x = 15$.

If we now add 1 to each side of the equation, the left-hand member will be a perfect square.

$$x^2 + 2x + 1 = 16,$$

or

$$(x + 1)^2 = 4^2.$$

Extracting the square root of each side,

$$x + 1 = \pm 4,$$

and $$x = 3 \text{ or } -5.$$

Check. $$9 + 6 - 15 = 0.$$
$$25 - 10 - 15 = 0.$$

We may sum up the foregoing process in the following

Rule. *Write the expression in the form* $ax^2 + bx + c = 0$.

Transpose the term not involving x *to the right-hand side of the equation.*

Divide both sides of the equation by the coefficient of x^2.

Add to both members the square of one half the coefficient of x, *thus making the left-hand member a perfect square.*

Rewrite the equation, expressing the left-hand member as the square of a binomial and the right-hand member in its simplest form.

Extract the square root of both members of the equation, not omitting the ± *sign in the right-hand member.*

Transpose the constant term, leaving x *alone on the left-hand side of the equation. The two values obtained on the right-hand side of the resulting equation by considering the* + *and* − *signs separately are the roots sought.*

Check by substituting the roots in the original equation, which should then reduce to an identity.

EXERCISES

Solve and check the following:

1. $x^2 + 5x - 6 = 0.$
2. $x^2 - 4x - 5 = 0.$
3. $x^2 + x - 1 = 0.$
4. $x^2 - \frac{2}{3}x + 3 = 0.$
5. $x^2 - \frac{1}{2}x = 2.$
6. $x^2 - \frac{3}{2}x - 1 = 0.$
7. $15x^2 - 17x + 4 = 0.$
8. $6x^2 + 7x - 5 = 0.$
9. $33x^2 + 25x = -2.$
10. $91x^2 + 120x - 91 = 0.$
11. $3x^2 - 28x - 55 = 0.$
12. $3x^2 - 2x = 1.$
13. $4x(5x + 1) + 2(7x + 2) = 0.$
14. $7x^2 + 9x - 610 = 0.$

15. $x^2 + 2x = 4.$

16. $3x^2 + 4x - 2 = 0.$

17. $9x^2 + 9x - 4 = 0.$

18. $11x^2 - 9x + 3 = 0.$

19. $x^2 - 5x + 2 = 0.$

20. $23x^2 + 13x - 7 = 0.$

21. $ax^2 + bx + c = 0.$

Solution. Transposing c,

$$ax^2 + bx = -c.$$

Dividing by a,
$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Adding $\left(\frac{b}{2a}\right)^2$ to both sides of the equation,

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a},$$

or
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking the square root of both sides,

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a},$$

or
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

22. $x^2 + mx - p = 0.$

23. $3x^2 + 2ax - 5a^2 = 0.$

24. $px^2 + x - 1 = 0.$

25. $ay^2 + by - c = 0.$

26. $my^2 - ny + 1 = 0.$

27. $bx^2 + cx + 3 = 0.$

28. $m^2y + my^2 + mn + ny = 0.$

29. $6m^2 + bmx = 2b^2x^2.$

30. $3x^2 = -2k^2 - 3kx\sqrt{3}.$

31. $(2x + c)^2 = 2x - c.$

101. Solution of quadratic equations by formula. In Exercise 20, above, the general quadratic equation, $ax^2 + bx + c = 0$, has been solved and the roots found to be

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (F)$$

The expression (F) is a general result and may be used to solve any quadratic equation which is in the standard form $ax^2 + bx + c = 0$, where a, b, and c may represent numbers, single letters, binomials, or any form of algebraic expression not involving x.

EXERCISES

Solve and check the following:

1. $3x^2 - 2x + 1 = 0$.

Solution. Here $a = 3$, $b = -2$, and $c = 1$.

Hence
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 3 \cdot 1}}{6}$$
$$= \frac{2 \pm \sqrt{4 - 12}}{6}$$
$$= \frac{1 \pm \sqrt{-2}}{3}.$$

Check.
$$3\left(\frac{1 \pm 2\sqrt{-2} - 2}{9}\right) - 2\left(\frac{1 \pm \sqrt{-2}}{3}\right) + 1 = -\frac{1}{3} \pm \frac{2\sqrt{-2}}{3} - \frac{2}{3} \mp \frac{2\sqrt{-2}}{3} + 1 = 0.$$

2. $x^2 - 2x - 17 = 0$.
3. $25x^2 - 10x - 1 = 0$.
4. $6x^2 + 7x - 3 = 0$.
5. $39x^2 - 83x = 56$.
6. $2x^2 + 5x = 3$.
7. $x^2 - 2x + 2 = 0$.
8. $7x^2 - 13x - 1 = 0$.
9. $2x^2 + 3x + 4 = 0$.
10. $x^2 - 7x + 2 = 0$.
11. $x^2 - x - 1 = 0$.
12. $2x^2 + x - 9 = 0$.
13. $\frac{3}{2}x^2 + \frac{1}{3}x - 1 = 0$.

102. Quadratic form. Any equation is in **quadratic form** if it may be written as a trinomial consisting of a constant term and two terms involving the unknown in question (or an expression which may be considered as the unknown), the exponent in one term being twice that in the other. By the **constant term** is meant the term not containing the unknown.

Thus $y - 2\sqrt{y} + 5 = 0$, $x^{\frac{1}{2}} + 3x^{\frac{1}{4}} - 12 = 0$, $x - 1 + \sqrt{x - 1} + 25 = 0$ are all in quadratic form. In the last equation the expression $x - 1$ is considered as the unknown.

It is often more convenient to replace by a single letter the lower power of the unknown in cases where the unknown is a polynomial with respect to which the equation is in quadratic form.

EXERCISES

Solve and check the following:

1. $x - \sqrt{x} - 12 = 0.$

Solution. Let $\sqrt{x} = y.$

Then $y^2 - y - 12 = 0,$

and $(y - 4)(y + 3) = 0,$

whence $y = 4 \text{ or } -3,$

and $\sqrt{x} = 4 \text{ or } -3.$

$x = 16 \text{ or } 9.$

Check.

$16 - \sqrt{16} - 12 = 0.$

$9 - \sqrt{9} - 12 \neq 0.$

Root 9 rejected.

2. $x - 2 - 12\sqrt{x - 2} = -35.$

Solution. Let $\sqrt{x - 2} = y.$

Then $y^2 - 12y + 35 = 0,$

whence $(y - 7)(y - 5) = 0,$

and $y = 7 \text{ or } 5.$

$\sqrt{x - 2} = 7 \text{ or } 5.$

$x - 2 = 49 \text{ or } 25.$

$x = 51 \text{ or } 27.$

Check.

$51 - 2 - 12\sqrt{51 - 2} = -35.$

$27 - 2 - 12\sqrt{27 - 2} = -35.$

3. $x + 2x^{\frac{1}{2}} + 1 = 0.$

4. $x - 2x^{\frac{1}{2}} + 1 = 0.$

5. $3y + \sqrt{y} = 2.$

6. $x - 6x^{-1} = 1.$

7. $3x^{\frac{3}{2}} + 8x + 4x^{\frac{1}{2}} = 0.$

HINT. Divide by $x^{\frac{1}{2}}$. This factor corresponds to the root $x = 0.$

8. $4x^{\frac{4}{3}} - x^{\frac{2}{3}} - 3 = 0.$

9. $3x^4 + 10x^2 + 8 = 0.$

10. $2x + 3x^{\frac{2}{3}} - 2x^{\frac{1}{3}} = 0.$

11. $\sqrt[3]{x^2} + 3\sqrt[3]{x} - 4 = 0.$

12. $ax^{2b} + bx^b + c = 0.$

13. $2x^6 - 3x^3 + 1 = 0.$

14. $25x^{12} + 20x^6 - 5 = 0.$

15. $x^{\frac{2}{5}} + x^{\frac{1}{5}} + \frac{1}{4} = 0.$

16. $9x^{\frac{4}{3}} + 6x^{\frac{2}{3}} + 4 = 0.$

17. $3x^2 - 2x + 2 - 5\sqrt{3x^2 - 2x + 2} + 6 = 0.$

18. $2x^4 + x^2 = 15.$

19. $x^6 + 2x^3 - 35 = 0.$

20. $x^8 - 21x^4 - 310 = 0.$

21. $x + 10\sqrt{x} + 21 = 0.$

22. $2x^2 - 3x^{\frac{6}{5}} - 9x^{\frac{2}{5}} = 0.$

23. $3(\sqrt{x}+1)^2 - 12 = 5(\sqrt{x}+1).$

24. $(x^2 - 3)(x^2 + 1) = 5.$

25. $(x^2 - 1)^2 + (x^2 + 1) = 74.$

26. $\left(x^{\frac{1}{3}} - 1\right)^2 + x^{\frac{2}{3}} = x^{\frac{1}{3}}.$

27. $\left(x^{\frac{1}{2}} + 2\right)^2 + x = 10x^{\frac{1}{2}}.$

28. $\left(x^{\frac{1}{3}} + 2\right)\left(x^{\frac{1}{3}} - 3\right) = -4.$

29. $(x + 3)^3 - (x + 2)^3 = 1.$

30. $(3x^2 - 5)^2 - (x^2 - 1)^2 = 4.$

31. $x^2 + x - \sqrt{x^2 + x + 1} = 5.$

32. $(x - 1)^{\frac{1}{2}} + (x - 1)^{-\frac{1}{2}} = \frac{10}{3}.$

33. $8(8x - 5)^3 + 5(5 - 8x)^6 = 85.$

34. $(7x + 1)^2 + 7(7x + 1) = -6.$

35. $x^4 - 4(a + b)x^2 + 16(a - b)^2 = 0.$

36. $6 - 5\sqrt{x - 1} = 6(x - 1).$

37. $(2x^2 + 4x - 3)^2 + 2x^2 + 4x = 9.$

38. $(x^2 - x + 2)^2 + (x - 3)(x + 2) = 64.$

39. $2x^2 = 9x + 3\sqrt{2x^2 - 9x + 4} - 4.$

40. $2(x - 4)^{\frac{2}{3}} + 4(x - 4)^{-\frac{2}{3}} = 9.$

41. $x^2 + \dfrac{2}{x^2} = 3.$

42. $\dfrac{x + 2}{x - 2} = x - 1.$

43. $x^{-\frac{1}{3}} + \dfrac{1}{x^{\frac{2}{3}}} = \dfrac{4}{9}.$

44. $a = x^2 + \left(\dfrac{b}{x}\right)^2 + b.$

45. $\sqrt{x} + 3 = \dfrac{5}{\sqrt{x} - 1}.$

46. $2 + x - \dfrac{2x^2}{\frac{1}{2}(2 + 2\sqrt{1 + x})^2} = 6.$

47. $3 - 9x + \dfrac{(3 - 9x)^2}{36} = -5.$

48. $(x^2 + 2)^{\frac{5}{2}} + \dfrac{3}{\sqrt{x^2 + 2}} = 4(x^2 + 2).$

49. $x^2 - 3x + 1 + (x + 2)(x - 5) = 47.$

50. $(2x + 3)^2 + 3(2x + 3) - 10 = 0.$

103. Problems solvable by quadratic equations. The principle of translating the problem into algebraic symbols, explained in section 55, should be observed here. The result should be verified in every case. It may happen that the problem implies restrictions that are not expressed in the equation to which the problem leads. In this case some of the solutions of the equation may not be consistent with the problem; for instance, when the variable stands for a number of men, fractional solutions should be rejected. If only such results are obtained, the problem is self-contradictory. Often negative solutions should be rejected, as when the result indicates a negative number of digits in a number. Imaginary or complex results (p. 178) in general mean that the conditions of the problem cannot be realized.

PROBLEMS

1. The product of $\frac{1}{3}$ and $\frac{1}{7}$ of a certain number is 2100. What is the number?

2. The product of two numbers is 5000. If the first number is as much less than 75 as the second number is greater than 75, find the two numbers.

3. The sum of the square roots of two numbers is $\sqrt{30}$. One of the numbers is 4 times the other. Find the numbers.

4. A fruit dealer had bananas to sell at $\frac{1}{4}$ as many cents per dozen as he had bananas in all. If he sold them all for $12, what was the price per dozen?

5. If the area of a rectangle which is twice as long as it is wide is 300 square feet, find its diagonal.

6. An automobile dealer sells cars so as to make a percentage profit equal to the selling price divided by 400. If his profits per car amount to $2500, find the selling price.

7. Two objects move on the sides of a right angle. If one is 100 feet from the vertex and approaching it at the rate of 10 feet per second, and the other object is 50 feet from the vertex and moving away from it at the rate of 15 feet per second, find when the two objects will be 150 feet apart.

8. Find a number such that twice its square shall exceed the number itself by 45.

9. Find two numbers such that their sum shall be 30 and their product equal to 221.

10. Two numbers differ by 2, while the sum of their squares is 130. What are the numbers?

11. By what number must both 15 and 12 be increased so that their product shall be increased by 280?

12. What numbers have a quotient of 3 and a product of 1125?

13. Two numbers are in the ratio of $2:1$. If each is increased by 3, the sum of their squares is 306. Find the numbers.

14. If 4 is divided by a certain number, the result is the same as though the number had been subtracted from 4. Find the number.

15. Separate 500 into two parts such that the sum of their reciprocals shall be equal to the reciprocal of 120.

16. The difference between a number and 30 times its reciprocal is 1. What is the number?

17. Twice the reciprocal of a certain number when added to 3 is equal to the number with its sign changed. What is the number?

18. The sum of the digits of a two-digit number is 9. If the number is multiplied by another number composed of the same two digits in reverse order, the product is 2430. Find the number.

19. The sum of two numbers is 200. The square root of one added to the other is equal to 68. Find the numbers.

20. The sum of two numbers is 5, and the sum of their cubes is 35. Find the numbers.

21. A flower bed is 6 yards by 4 yards. It is surrounded by a gravel path of uniform width having an area equal to the area of the flower bed. Find the width of the gravel path.

22. Two motorists start from points 130 miles apart and travel toward each other. The speed of one motorist is 30 miles per hour, while the speed of the other is 33 more than the number of hours before they meet. How far does each travel before they meet? What length of time elapses before they meet?

23. A cart contains sacks of potatoes. There are one less than $\frac{1}{20}$ as many sacks as the average number of potatoes per sack. If the number of sacks is increased 25%, the number of potatoes per sack must be reduced by 20 in order that the total number of potatoes in the cartload will be the same. What is the number of potatoes per sack?

24. A dealer buys material for a certain sum of money. He sells the same material for $2400. After deducting a fixed charge of 6% on the cost for handling the goods, he finds that he has made a percentage of net profit equal to 6 less than $\frac{1}{100}$ of the cost price. What did he pay for the material?

25. A man put $1000 in the savings bank at interest compounded annually. At the end of each year thereafter he added $100 to his principal in addition to his interest. At the end of the second year he had $1285.60. Find the interest rate.

26. Two men can do a piece of work in 12 days, working together. It takes B 10 days longer than A to do the work alone. Find how long it would take A to do the work alone.

27. A tank can be emptied by means of two outlets in $1\frac{3}{7}$ hours. If one pipe can empty the tank in 3 hours less time than the other pipe, find the length of time required by each.

28. A tank is filled by one pipe and can be emptied by means of another pipe. When both pipes are open the tank can be emptied in $1\frac{1}{2}$ hours. If the tank can be emptied in 2 hours less time than it takes to fill it, find the length of time needed to fill and to empty the tank.

29. A party spent $18 for meals in a restaurant. If there had been 6 more people in the party, the average amount spent per person would have been 50 cents less if the total amount of the bill had remained the same. How many were in the party and what was the average amount spent per person?

30. A farmer has 260 head of cattle divided among several herds. If the number of herds were increased by 7, the number of cattle per herd would have to be reduced by 7. How many herds did the farmer have?

31. A party pays $16 for hotel accommodations. If there had been one person less in the party, the cost would have been $8 higher per person. Find the number of persons in the party.

32. A grocer sells eggs for $7.50. If he had had 5 dozen eggs less, he would have had to increase his price 25 cents per dozen in order to receive the same amount. How many dozen eggs did he have?

33. A man purchases paper clips for $1.50. If he had received 500 fewer clips, they would have cost him $\frac{1}{20}$ cent more per clip. How many did he purchase?

34. It took a number of men as many days to do a certain piece of work as there were men working on the job. If there had been 3 more men, the number of days required would have been 4. Find the number of men doing the work.

35. A workman does a number of small jobs in 5 hours and 50 minutes. If he had done 10 jobs less in the same time, the average time per job would have been 4 minutes greater. Find the number of jobs done.

QUADRATICS AND BEYOND

CHAPTER X

PROPERTIES AND GRAPHS OF QUADRATIC EQUATIONS

104. Relationship of roots and factors. In this and the following sections we will prove several theorems concerning quadratic equations. Similar theorems will later be proved in general for equations of higher degree.

Theorem. *If r is a root of the equation*

$$ax^2 + bx + c = 0, \tag{1}$$

then $x - r$ is a factor of its left-hand member.

The fact that r is a root of the equation is equivalent to the assertion of the truth of the expression

$$ar^2 + br + c = 0.$$

This is an identity by the definition of the root of an equation as given in section 53.

Hence

$$\begin{aligned} ax^2 + bx + c &= ax^2 + bx + c - (ar^2 + br + c) \\ &= a(x^2 - r^2) + b(x - r) \\ &= (x - r)(ax + ar + b). \end{aligned} \tag{2}$$

That is, $(x - r)$ is one factor of $ax^2 + bx + c$. Another factor, from which another root can be found, is given in (2).

The converse of the foregoing theorem, namely, the fact that if $(x - r)$ is a factor of a quadratic equation, r is a root, has already been explained, and used (section 99) as one of the methods of solving quadratic equations.

EXERCISES

Form equations which have the following roots:

1. 7, 3.

Solution. Since 7 and 3 are roots, $(x-7)$ and $(x-3)$ are factors, and the left-hand member of the equation becomes $x^2 - 10x + 21$. The equation is therefore

$$x^2 - 10x + 21 = 0.$$

2. 3, 2.
3. 2, -1.
4. 5, -4.
5. $\sqrt{3}$, $-\sqrt{3}$.
6. $2\sqrt{8}$, $3\sqrt{2}$.
7. $2\sqrt{-1}$, $-2\sqrt{-1}$.
8. $5\sqrt{32}$, $2\sqrt{2}$.
9. $2+\sqrt{3}$, $2-\sqrt{3}$.
10. $7-\sqrt{-2}$, $7+\sqrt{-2}$.
11. $14-\sqrt{3}$, $14+\sqrt{3}$.
12. $a-\sqrt{b}$, $a+\sqrt{b}$.
13. $a+\sqrt{b-c}$, $a-\sqrt{b-c}$.

105. Number of roots of a quadratic equation.

Theorem. A quadratic equation has only two roots.

Given the equation $ax^2 + bx + c = 0$ with the roots r and s, to prove that the equation has no other root, such as t, which is distinct from r and s.

Since r and s are roots of the equation, $(x-r)$ and $(x-s)$ are factors. Thus our equation may be written in the form

$$a(x-r)(x-s) = 0. \tag{1}$$

But if t is a root, it must satisfy (1), that is,

$$a(t-r)(t-s) = 0.$$

But, in order that any product of numeric factors should vanish, one of the factors must vanish. Thus either $a = 0$, or $t - r = 0$, or $t - s = 0$. But, by hypothesis, $t \neq r$, and $t \neq s$, so that the last two factors cannot vanish. Thus $a = 0$. This would, however, reduce our equation to the form of a linear equation, which is contrary to our hypothesis that the equation is quadratic. Hence the assumption that we have a third distinct root leads to a contradiction.

NOTE. This theorem taken with section 104 is equivalent to the statement that a quadratic equation can be factored in one and only one way. Thus, if

$$ax^2 + bx + c = a(x - r)(x - s),$$

we cannot find numbers t and v distinct from r and s, such that

$$ax^2 + bx + c = a(x - t)(x - v),$$

for then the equation would have roots distinct from r and s, which is contrary to the assumption.

106. Nature of the roots of a quadratic equation. We have already seen, in section 101, that the general quadratic equation

$$ax^2 + bx + c = 0 \tag{1}$$

has as roots

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{2}$$

and

$$s = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{3}$$

These expressions afford an immediate arithmetic means of determining the nature of the roots of the given equation when a, b, and c have numeric values and $a \neq 0$. In fact, an inspection of the value of $b^2 - 4ac$ in (2) and (3) is sufficient to determine the nature of the roots of (1).

I. When $b^2 - 4ac$ is positive, the roots are real and distinct.

II. When $b^2 - 4ac$ is positive and a perfect square, the roots are real, distinct, and rational.

III. When $b^2 - 4ac = 0$, the roots are real and equal. In this case $r = s = -\frac{b}{2a}$.

IV. When $b^2 - 4ac$ is negative, the roots are imaginary.

The converses of these four cases are also true. For instance, if the roots of (1) are imaginary, it is clear from (2) and (3) that $b^2 - 4ac$ must be negative.

The expression $D = b^2 - 4ac$ is called the *discriminant* of the equation $ax^2 + bx + c = 0$.

EXERCISES

Determine the nature of the roots of the following equations without solving:

1. $2x^2 - 3x - 2 = 0.$

Solution. $D = (-3)^2 - 4(2)(-2) = 9 + 16 = 25.$ Since 25 is positive and a perfect square the roots are real, distinct, and rational.

2. $x^2 - 3x + 4 = 0.$
3. $9x^2 + 2x + 1 = 0.$
4. $3x^2 + x - 1 = 0.$
5. $x^2 - x - 1 = 0.$
6. $x^2 + x + 1 = 0.$
7. $3x^2 + 2x - 1 = 0.$
8. $x^2 - 2x - 3 = 0.$
9. $13x^2 - 5x + 8 = 0.$
10. $2x^2 + 2x + 4 = 0.$
11. $3x^2 - 7x + 2 = 0.$

Determine the real values of k so that the roots of the following equations may be real and equal:

12. $(k-1)x^2 - (k+2)x + 4 = 0.$

Solution. Here $k - 1 = a$, $-(k+2) = b$, and $4 = c$.

Thus
$$\begin{aligned} D &= (k^2 + 4k + 4) - 4 \cdot 4(k-1) \\ &= k^2 - 12k + 20. \end{aligned}$$

Since the roots of an equation are real and equal when and only when the discriminant is equal to zero, the required values of k make $D = 0$ and are the roots of the equation

$$k^2 - 12k + 20 = 0.$$

Solving by factoring, $(k-10)(k-2) = 0,$

and $k = 10$ or 2.

Check. Substituting in the original equation for $k = 10$, we get

$$9x^2 - 12x + 4 = (3x-2)^2,$$

and for $k = 2$, we get $x^2 - 4x + 4 = (x-2)^2.$

13. $x^2 + kx + 4 = 0.$
14. $x^2 + 6x + k = 0.$
15. $kx^2 - 24x + 9 = 0.$
16. $4x^2 + 3kx + 9 = 0.$
17. $x^2 + (k-2)x + 1 = 0.$
18. $(2k+5)x^2 - kx + 1 = 0.$
19. $(k-31)x^2 + 136x + 289 = 0.$
20. $x^2 - 2kx + 3 = 0.$
21. $kx^2 + 3kx + k + 5 = 0.$
22. $(k+21)x^2 - 44x + 4 = 0.$
23. $kx^2 + 7kx + 49 = 0.$
24. $x^2 + (96-k)x + 2k = 0.$
25. $(k-1)x^2 + (16-k)x + 1 = 0.$
26. $x^2 - 2kx + 2 = 0.$
27. $x^2 - 2k - 1 = 0.$
28. $(k+6)x^2 - (k-2)x + 1 = 0.$
29. $x^2 - kx + k = 0.$
30. $4x^2 - (k-6)x + 2k + 21 = 0.$
31. $x^2 + (k-a)x + a^2 = 0.$
32. $a^2x^2 + kax + 2k + 5 = 0.$

107. The sum and difference of roots. Let r and s be the roots of

$$x^2 + bx + c = 0. \qquad (1)$$

Then, by section 104, $(x - r)$ and $(x - s)$ are factors, and their product, $x^2 - (r + s)x + rs$, is exactly the left-hand member of (1). Consequently the equation

$$x^2 + bx + c = x^2 - (r + s)x + rs$$

is true for all values of x. Hence

$$-(r + s) = b, \qquad (2)$$

and

$$rs = c. \qquad (3)$$

We may state these facts in the following

Theorem. The coefficient of x in the equation $x^2 + bx + c = 0$ is equal to the sum of its roots with their sign changed.*

The constant term is equal to the product of the roots.

EXERCISES

1. Prove the above theorem by using the formula for the roots of an equation in terms of its coefficients as given in section 101.

Form equations whose roots are the following:

2. 3, 2.
3. 7, -3.
4. 11, 13.
5. -3, -4.
6. $\frac{1}{2}$, $\frac{3}{4}$.
7. $\frac{1}{3}$, $\frac{1}{7}$.
8. $7 - \sqrt{3}$, $7 + \sqrt{3}$.
9. $-\sqrt{5}$, $\sqrt{5}$.

Find the value of the literal coefficient in each of the following equations:

10. $x^2 - 5x + c = 0$, when one root is 4.

Solution. Let r be the remaining root.

Then by (2), $\quad -(r + 4) = -5,$

or $\quad r = 1.$

By (3), $\quad c = 1 \cdot 4 = 4.$

---

* We should for the present exclude the case where $b^2 - 4ac < 0$, since the roots r and s are then imaginary and we have not as yet defined what we mean by the sum or the product of imaginary numbers. We shall see later that the theorem is also true in this case.

11. $x^2 - bx - 6 = 0$, when one root is 2.

12. $x^2 + x + c = 0$, when one root is 4.

13. $x^2 + bx + 3 = 0$, when one root is 3.

14. $x^2 + 10x + c = 0$, when one root is -11.

15. $x^2 + 10x + c = 0$, when one root is -3.

16. $x^2 + 10x + c = 0$, when one root is -5.

17. $x^2 - bx + 2 = 0$, when one root is -1.

18. $x^2 - bx + 1 = 0$, when the roots are equal.

19. $x^2 + bx - 4 = 0$, when the roots are equal in value but opposite in sign.

20. $x^2 + 6x - c = 0$, when one root is twice the other.

21. $x^2 - bx - 4 = 0$, when one root is 4 greater than the other.

22. $x^2 - x + c = 0$, when one root is 17 greater than the other.

23. $x^2 + 5x + c = 0$, when one root is 1 greater than the other.

24. $x^2 + bx + 15 = 0$, when the difference between the roots is 2.

25. $x^2 - 6x + c = 0$, when the difference between the roots is 1.

26. $x^2 + 17x + c = 0$, when the difference between the roots is 3.

108. The special quadratic equation $ax^2 + bx = 0$. When in the quadratic equation

$$ax^2 + bx + c = 0, \tag{1}$$

$c = 0$, we can always factor the equation into

$$ax^2 + bx = x(ax + b) = 0,$$

or

$$x\left(x + \frac{b}{a}\right) = 0.$$

Thus the roots are $$r = 0, \quad s = -\frac{b}{a}.$$

Conversely, if $r = 0$ is a root, then, by section 104, $(x - 0)$, or x, is a factor and the equation can have no constant term.

This affords the following

Theorem. A quadratic equation has a root equal to zero when and only when the constant term vanishes.

We show in a similar manner that both roots of equation (1) are zero when and only when $b = c = 0$.

EXERCISES

1. Prove the above theorem by using the formula for the roots of a quadratic equation in terms of its coefficients (section 101).

For what real values of k do the following equations have one root equal to zero?

2. $x^2 - 3x + k - 1 = 0.$
3. $x^2 + 5x - k^2 + 5 = 0.$
4. $x^2 + 2x + k^2 + 4 = 0.$
5. $3x^2 + 2x + 5k^2 - 3k = 0.$
6. $2x^2 + 3kx + k^2 + 3k + 2 = 0.$
7. $7x^2 - kx + 2k^2 + 9k - 5 = 0.$

For what real value of k will both roots of the following equations vanish?

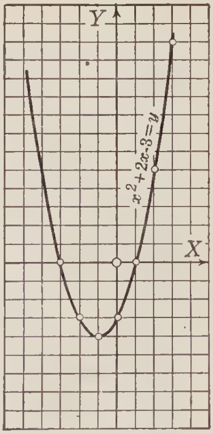
8. $x^2 + 2kx - 3k^2 = 0.$
9. $\frac{x^2}{k^2} + 7x - 8 = 0.$
10. $x^2 + (k - 1)x + k = 0.$
11. $x^2 + (k^2 + 3)x - 3k = 0.$
12. $(k + 2)x^2 - 3(k^2 - 4)x + 2k^2 + 5k + 2 = 0.$
13. $(k + 1)x^2 + 2(k - 1)x + k^2 - 1 = 0.$

109. Graph of a quadratic equation. Following the method of plotting an equation suggested in section 57, let us graph the equation

$$y = x^2 + 2x - 3. \qquad (1)$$

To this end we shall let x take on various integral values, and compute the corresponding values of y, thus determining the coördinates of points on the curve which represents the equation.

x	-3	-2	-1	0	1	2	3
y	0	-3	-4	-3	0	5	12



Plotting the points afforded by this computation, we obtain the accompanying graph.

It is seen that the roots of the equation $x^2 + 2x - 3 = 0$ are the numbers $+1$ and -3. These values substituted for

x make the expression equal to zero, that is, they are its roots. Let us now consider equation (1) above. The values of x, $+1$ and -3, are the values for which y becomes zero. This is another way of saying that the corresponding points are to be found on the x-axis of the graph. Hence we may infer that, had we plotted equation (1) without any knowledge of the roots of the right-hand member, the x-value of the points where the curve crossed the x-axis would have shown us the value of the roots.

110. Graphs of the general quadratic equation. We will now discuss in more general terms the relation between the quadratic equation and its graph.

Let
$$y = ax^2 + bx + c, \tag{1}$$
where, as usual, a, b, and c represent integers and a is positive.

If we let x take on various values, y will have corresponding values and we may plot the equation as in section 109. A root of the quadratic equation
$$ax^2 + bx + c = 0 \tag{2}$$
is a number which, substituted for x, satisfies the equation, that is, gives the value $y = 0$ in (1). Thus the points on the graph of (1) which represent the roots of the equation (2) are the points for which $y = 0$, that is, where the curve crosses the x-axis. The numeric value of the roots is the measure of the distance along the x-axis from the origin to the points where the curve cuts the axis. Since this distance is always a real distance, only real roots are represented in this manner.

Theorem. If the graph of (*1*) *has no point in common with the x-axis, the equation* (*2*) *has imaginary roots, and conversely.*

Every equation of the form (2) has two roots, either real or imaginary (section 105). If the graph of (1) has no point in common with the x-axis, there is no *real* value of x for which $y = 0$, that is, no real root of (2). The roots must then be imaginary.

Conversely, if (2) has only imaginary roots, there is no real value of x which satisfies it, that is, which makes $y = 0$ in (1). Thus the curve has no point in common with the x-axis.

This suggests the following

Principle. Non-intersections of graphs correspond to imaginary or infinite-valued solutions of equations.

111. Form of the graph of a quadratic equation. Consider the equation

$$y = 3x^2 + 2x - 5. \qquad (1)$$

By substituting for x a very large positive or negative number, say $x = \pm 100$, y is large positively. Thus for values of x far to the right or left, the curve lies far above the x-axis. If we assign to y a certain value, say $y = -5$, we can find the corresponding values of x by solving a quadratic equation. Thus in (1), let $y = -5$.

Then $$-5 = 3x^2 + 2x - 5,$$

or $$3x^2 + 2x = 0.$$

The roots are $$r = -\tfrac{2}{3}, \quad s = 0.$$

Hence the points $(-\frac{2}{3}, -5)$ and $(0, -5)$ are on the curve (section 57). That is, if we go down five units on the y-axis, the curve is to be found two thirds of a unit to the left and also on the y-axis. If in (1) we let $y = 3$, the corresponding values of x are $\frac{4}{3}$ and -2, which means that the curve meets a line parallel to the x-axis and three units above it at these points. The question now arises, Where is the bottom (or top) of the loop of the curve? This lowest (or highest) point has as its value of y that number to which correspond *equal* values of x. Hence we must determine for what value of y equation (1), that is, the equation

$$3x^2 + 2x + (-5 - y) = 0,$$

has equal roots. Comparing with the equation $ax^2 + bx + c = 0$ (section 106), we have

$$3 = a, \quad 2 = b, \quad -5 - y = c.$$

Thus the condition $b^2 - 4ac = 0$ becomes

$$4 - 4 \cdot 3(-5 - y) = 0,$$

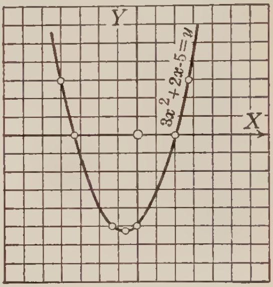
or

$$y = -\frac{4 + 60}{12} = -\frac{16}{3} = -5\tfrac{1}{3}.$$

Substituting this value of y in (1), we get $-\frac{1}{3}$ as the corresponding value of x.

This gives a single value of y for which the values of x are equal; hence the graph of (1) is a single festoon as in the figure.

x	1 or $-1\frac{2}{3}$	0 or $-\frac{2}{3}$	$\frac{4}{3}$ or -2	$-\frac{1}{3}$
y	0	-5	3	$-5\frac{1}{3}$



If we take the general equation

$$ax^2 + bx + c = y,$$

we find, similarly, that the bottom of the loop is at a point whose ordinate is

$$y = -\frac{b^2 - 4ac}{4a} = -\frac{D}{4a}.$$

Thus we see again that if the discriminant is negative the graph is entirely above the x-axis and both roots are imaginary (sections 106 and 110), since the ordinate of the lowest point of the loop is positive. If the discriminant is positive, the graph cuts the x-axis and both roots are real.

The results of this section enable us to determine a value of y for the lowest point of the curve, and hence to show from the graph whether the equation has imaginary roots. It is not difficult to show that, while this method of finding high and low points is satisfactory in the case of quadratic equations, it is not general and breaks down before higher-degree equations. A method of determining the *maxima* and *minima* of the graph of a higher-degree equation is beyond the scope of this book. The interested student will find the method described in textbooks of analytic geometry.

EXERCISES

Plot the following equations and determine by measurements the roots in case they are real. Find in each case the lowest point on the loop.

1. $x^2 + 2x + 2 = y$.
2. $x^2 - 5x + 4 = y$.
3. $x^2 + 3x + 2 = y$.
4. $x^2 - 5x + 6 = y$.
5. $x^2 - 5x - 6 = y$.
6. $2x^2 - 3x + 4 = y$.
7. $x^2 + 3x + 1 = y$.
8. $3x^2 + 3x - 1 = y$.
9. $2x^2 + 3x + 3 = y$.
10. $x^2 + x + 1 = y$.
11. $x^2 - x - 1 = y$.
12. $x^2 - x + 1 = y$.
13. $x^2 + x - 1 = y$.
14. $4x^2 + 3x - 5 = y$.
15. $6x^2 + 13x + 6 = y$.
16. $7x^2 + 3x - 4 = y$.

17. What is the characteristic feature of the graph of an equation one of whose roots is zero?

18. What is the characteristic feature of the graph of an equation whose roots are equal?

112. Degeneration of the quadratic equation. The equation

$$ax^2 + bx + c = 0$$

has the roots
$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and
$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We wish to find the effect on the roots x_1 and x_2 of letting a become very small. If we let a approach 0, then x_1 approaches an expression of the form $\frac{0}{0}$, which must always be avoided. Let us, then, rationalize the denominators. We now get

$$x_1 = \frac{4ac}{2a(-b - \sqrt{b^2 - 4ac})} = \frac{2c}{-b - \sqrt{b^2 - 4ac}},$$

and
$$x_2 = \frac{4ac}{2a(-b + \sqrt{b^2 - 4ac})} = \frac{2c}{-b + \sqrt{b^2 - 4ac}}.$$

As a approaches 0, evidently $b^2 - 4\,ac$ approaches b^2, x_1 approaches $-\frac{c}{b}$, and x_2, since its denominator becomes very small, increases without limit, that is, becomes infinite. Thus the quadratic equation approaches a linear equation when a approaches 0, and one of its roots disappears since it has increased in value beyond any finite limit. The loop-shaped graph of the quadratic equation must then approach a straight line as a limit when a approaches 0. This is made clear from the following figure, where a has the successive values 1, $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{50}$, 0.

In the figure the curves represent the following equations:

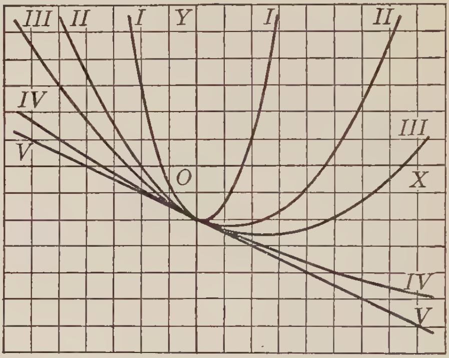
$$x^2 - \frac{x}{2} - 2 = y. \quad \text{(I)}$$

$$\frac{x^2}{5} - \frac{x}{2} - 2 = y. \quad \text{(II)}$$

$$\frac{x^2}{10} - \frac{x}{2} - 2 = y. \quad \text{(III)}$$

$$\frac{x^2}{50} - \frac{x}{2} - 2 = y. \quad \text{(IV)}$$

$$-\frac{x}{2} - 2 = y. \quad \text{(V)}$$



In a similar manner we can show that when in the equation $bx + c = 0$, b approaches 0 as a limit, the root of the linear equation becomes infinite.

EXERCISES

What real values must k approach as a limit in order that one root of each of the following equations may become infinite?

1. $kx^2 + 3\,x - 2 = 0.$
2. $(k^2 - 4)x^2 - x + 3 = 0.$
3. $(kx + 1)^2 - (x + 1)^2 = (k - x)^2.$
4. $k^2 + 3\,k^2x^2 - (x + 1)^2 + 3 = 0.$

5. $\sqrt{6\,kx+2}+\sqrt{2\,k+3}=\sqrt{2\,kx-1}$.

6. $\sqrt{k-x}+\sqrt{k+x}=\sqrt{2\,k+x}$.

7. $\frac{x+1}{k+1}-\frac{k-1}{x-1}+\frac{k}{x}=0$.

8. $\sqrt{\frac{k+x}{k-x}}+\sqrt{\frac{k-x}{k+x}}=\sqrt{\frac{3\,k+1}{k^2}}$.

9. $\frac{(x+1)^2}{(x-1)^2}+\frac{k-1}{k+1}+\frac{k}{8}=0$.

10. $(k^2-9)x^2+(k+3)x+k^2-2\,k-15=0$.

What real values must k and m approach as a limit in order that both roots of the following equations may become infinite?

11. $kx^2+mx-3=0$.

12. $(6\,k-m)x^2-mx+3=0$.

13. $7\,kx^2+(3\,k-m+2)x=2\,mx^2+7$.

14. $(k+2)x^2-(3\,k+m-1)x+2=0$.

15. $(k^2+3\,k+2)x^2-7(k^2-8\,km-20)x+6=0$.

16. $(k+m)x^2-(k+m)-3=x^2+2\,x$.

17. $(k-m+3)x^2+2(k+m)x-4=0$.

18. $(7\,k+3\,m-1)x^2-2=(13\,k-5+m)x$.

CHAPTER XI

SIMULTANEOUS QUADRATIC EQUATIONS IN TWO VARIABLES

113. Solution of simultaneous quadratics. A single equation in two variables, as $x^2 + y^2 = 7$, is satisfied by many pairs of values, as $(0, \sqrt{7})$, $(1, \sqrt{6})$, $(2, \sqrt{3})$, and so on, though there are at the same time numberless pairs of values which do not satisfy it, as (1, 1), (2, 2), and (2, 3). Thus the condition that (x, y) satisfy a single quadratic equation imposes a considerable restriction on the values that x and y may assume. If we still further restrict the value of the pair of numbers (x, y) so that they also satisfy a second equation, the number of solutions is correspondingly reduced. The problem of solving two simultaneous equations consists in finding the pairs of numbers that satisfy them both.

114. Solution by substitution. In this method of solution the restriction imposed on (x, y) by one equation is imposed on the variables in the other equation by substitution.

EXAMPLE

Solve
$$2x^2 + y^2 = 1, \qquad (1)$$
$$x - y = 1. \qquad (2)$$

Solution. Equation (2) states that $x = 1 + y$. Thus our desired solution is such a pair of numbers that (1) is satisfied and at the same time x is equal to $1 + y$.

If we substitute in (1) $1 + y$ for x, we are imposing on its solution the restriction implied by (2).

Thus
$$2(1 + y)^2 + y^2 = 1,$$
or
$$3y^2 + 4y + 1 = 0.$$
The roots are
$$y = -1, -\tfrac{1}{3}.$$

Corresponding to $y = -1$, we get from (2), $x = 0$.
Corresponding to $y = -\frac{1}{3}$, we get from (2), $x = \frac{2}{3}$.
Thus the solutions are $(0, -1)$ and $(\frac{2}{3}, -\frac{1}{3})$.

Check. Substituting $(0, -1)$ in (1) and (2), we obtain

$$2 \cdot 0 + (-1)^2 = 1,$$
$$0 - (-1) = 1.$$

Substituting $(\frac{2}{3}, -\frac{1}{3})$ in (1) and (2), we obtain

$$2 \cdot \tfrac{4}{9} + \tfrac{1}{9} = \tfrac{9}{9} = 1,$$
$$\tfrac{2}{3} - (-\tfrac{1}{3}) = 1.$$

EXERCISES

Solve the following and check as directed by the teacher:

1. $x - y = 4$, $xy = 5$.
2. $x + y = 4$, $xy = 4$.
3. $x + y = 1$, $x^2 + y^2 = 3xy$.
4. $x + y = m$, $x^2 + y^2 = nxy$.
5. $7x + y = 15$, $xy + x = 4$.
6. $x + y = a$, $x^2 - y^2 = b$.
7. $x^2 + y^2 = 41$, $3x + 2y = 23$.
8. $x^2 - y^2 = 21$, $3x - y = 13$.
9. $7x + 5y = 12$, $x^2 + y^2 = 2$.
10. $xy = 6$, $3x + 5y = 21$.
11. $\frac{x}{y} = \frac{1}{9}$, $\frac{x}{3} = \frac{3}{y}$.
12. $\frac{x}{a} = \frac{c}{y}$, $bx = dy$.
13. $\frac{x^2}{y^2} = \frac{a^2}{b^2}$, $a + x = y - b$.
14. $3x^2 + y = 4xy$, $7x - 3y = 4$.
15. $x^2 + xy - y^2 = 1$, $x + 3y = 4$.
16. $(x + y)(2x + 3y) = 180$, $x - 2y = 3$.
17. $3x^2 - 12y = 2y^2 - x$, $2x - 7y = -18$.
18. $x^2 + y = y^2 + x - 10$, $\frac{x}{y} = \frac{1}{2}$.
19. $ax - by = cy$, $x^2 + xy = y^2$.
20. $ax - by = cy$, $a^2x^2 + 2acxy + c^2y^2 = m^2$.
21. $x^2 + 2xy + y^2 = 3y + 3x$, $3x - y = 5$.

22. $ax^2 + (a - b)xy - by^2 = 2\,b(a^2 + 2\,ab - b^2),$
$\frac{x + y}{x - y} = \frac{b}{a}.$

23. $3\,x^2 - 7\,xy + 2\,y^2 - 8\,y + 3\,x = -30,$
$x + y = 5.$

24. $2(x + 1)^2 - 5(y - 1)^2 = 52,$
$x - y = 6.$

25. $\frac{3\,x + y - 1}{2\,x + 3\,y + 6} = \frac{1}{2},$
$2\,x^2 - 3\,xy + y^2 = 8.$

26. $x^2 - y^2 = a^2,$
$\frac{x}{y} = \frac{a}{b}.$

27. $x^2 + y^2 = 4,$
$\frac{x + y}{x - y} = 1.$

28. $\frac{x^2 + 2\,xy - y^2}{y^2 + 2\,xy - x^2} = 7,$
$x + y = 3.$

29. $\frac{1 - x - x^2}{1 - y - y^2} = \frac{19}{5},$
$2\,x - 3\,y = 2.$

30. $xy + 3 = 7(3\,x - y),$
$\frac{x}{y} = \frac{1}{2}.$

31. $\frac{2\,x - y}{y} + \frac{x}{x + y} = -1,$
$x + y = -2.$

32. $\frac{7\,x - 5\,y + 2}{3\,x - 2\,y - 3} + \frac{x - 2\,y}{2\,x - y} = \frac{43}{16},$
$7\,x - 3\,y = -27.$

33. $x(x + y) + 2\,x = 28,$
$\frac{y}{x + y} = -2\tfrac{1}{2}.$

34. $\frac{5}{x + 2} + \frac{3}{y - 1} = 4\tfrac{2}{3},$
$\frac{7}{x - 2} = -\frac{14}{y}.$

35. $\frac{2}{x} + \frac{3}{y} = 5,$
$7(x + y) = 2(y + 6\,x).$

115. Plotting quadratic equations in two variables. When neither x nor y is already alone on one side of the equation, the equation should be solved for y (or x) and the resulting equation plotted as shown in sections 57 and 109. It should be noted that when a root is extracted two values of the variable y may correspond to a single value of x.

EXERCISES

Plot the following:

1. $4\,x^2 + 9\,y^2 = 81.$

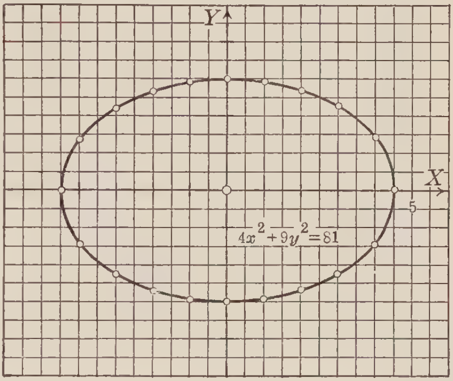
Solution. $$y^2 = 9 - \frac{4\,x^2}{9},$$

and $$y = \pm\sqrt{9 - \frac{4\,x^2}{9}}.$$

Assuming the various integral values for x, we obtain the following table and the corresponding graph:

x	0	± 1	± 2	± 3	± 4	± 5
y	± 3	± 2.9	± 2.7	± 2.2	± 1.4	Imaginary

In this example, when x is greater than 5 or less than -5, y is imaginary. Thus none of the curve is found outside a strip $x = \pm 5$.



To find exactly where the curve crosses the x-axis, the equation may be solved for x and the value of x corresponding to $y = 0$ found. Thus

$$x = \pm\sqrt{\frac{81 - 9y^2}{4}}.$$

If $y = 0$, $x = \pm\sqrt{\frac{81}{4}} = \pm 4\frac{1}{2}$. This point is included in the graph.

Any equation of the form $ax^2 + by^2 = c$ has a graph of the same general shape as the adjacent curve. Such a curve is called an ellipse. The circle may be looked on as a special case of the ellipse. When in the equation for an ellipse $a = b = 1$, that is, when the equation for the ellipse reduces to $x^2 + y^2 = c$, the equation is represented by a circle.

2. $xy = 4$.

Solution. $y = \frac{4}{x}$.

Form the table for integral values of x.

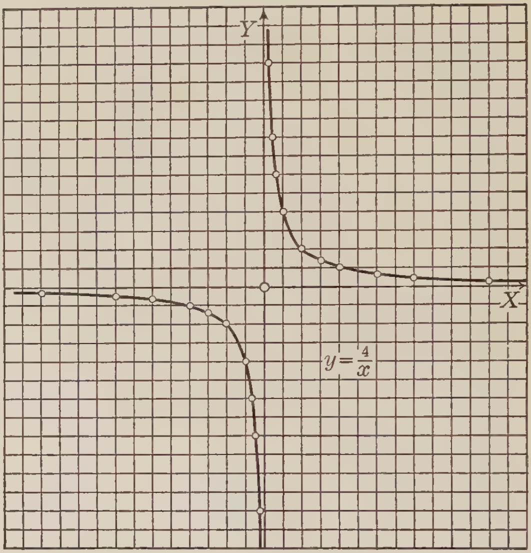
x	-12	-8	-6	-4	-3	-2	-1
y	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{2}{3}$	-1	$-\frac{4}{3}$	-2	-4

x	12	8	6	4	3	2	1
y	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{4}{3}$	2	4

Since this table does not give us any idea of the curves between $+1$ and -1, we supplement the table by assuming fractional values of x.

x	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$
y	-12	-8	-6	6	8	12

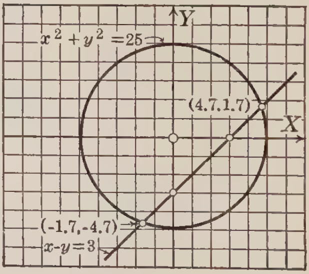
Any equation of the form $xy = a$ has a curve of the general form of the adjacent curve. This is called the **hyperbola.** It always consists of two parts or branches which are not connected. Both of these two branches are considered part of the same curve because the same equation yields both branches as parts of its graph. The hyperbola and other curves whose equations are quadratic are known as "conic sections."



3. $x^2 + y^2 = 25$.
4. $36\,x^2 + 4\,y^2 = 144$.
5. $xy = 15$.
6. $x^2 - y^2 = 16$.
7. $2\,x^2 + 3\,y^2 = 4$.
8. $x^2 + y^2 = 9$.

116. Graphical representation of simultaneous quadratic equations. Every equation that we have considered may be represented graphically by plotting.

The common values of simultaneous equations are represented by the points of intersection of the corresponding graphs.



Thus the equations

$$x^2 + y^2 = 25,$$
$$x - y = 3$$

have the solutions

$$x = \frac{3 \pm \sqrt{41}}{2},$$
$$y = \frac{-3 \pm \sqrt{41}}{2},$$

or

$$x = 4.7, \quad -1.7,$$
$$y = 1.7, \quad -4.7.$$

These equations have as their graph the preceding figure. It is observed that the coördinates of the points of intersection of the circle and the line are the same as the sets of roots found by algebraic solution. This must always be the case for real roots, since the values of x and y that satisfy both equations must lie on the graphs of both equations.

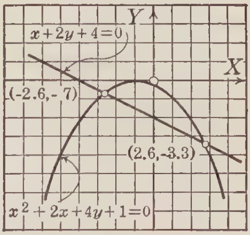
Hence, if the equations had been plotted without any previous knowledge of the roots, their value could be read from the finished graph to a considerable degree of accuracy. This is called the graphical solution of simultaneous equations. The graphical solution is useful when the curves seem to be easier to plot than to solve, and when only a slight degree of accuracy is essential. It is also often useful as an approximate check on the more exact algebraic solution.

The equations

$$x^2 + 2x + 4y + 1 = 0,$$
$$x + 2y + 4 = 0$$

have the solutions

$$x = \pm\sqrt{7} = \pm 2.6,$$
$$y = -2 \mp \frac{\sqrt{7}}{2} = -3.3 \text{ or } -.7.$$



An inspection of the graph shows us that, while the algebraic solution gives us quite accurate values of the roots, the graphical solution cannot here be relied upon farther than the first decimal place.

EXERCISES

Solve the following systems of equations graphically. Estimate roots to one decimal place.

1. $x^2 + 3xy + y^2 = 8$,
   $x + y = 3$.
2. $x^2 + y^2 = 5$,
   $x - y = -1$.
3. $xy = 6$,
   $x + y = 5$.
4. $2x^2 + 4y^2 = 22$,
   $2x - 3y = 3$.
5. $x^2 + 3xy - y^2 = 4$,
   $x - 3y = 7$.
6. $3x^2 + 9y^2 = 14$,
   $x + y = 1$.
7. $x^2 + y^2 = 625$,
   $x + y = 25$.
8. $xy = 12$,
   $x + 3y = 13$.
9. $x^2 - y^2 = 7$,
   $x + 3y = 13$.
10. $x^2 + 3xy + y^2 = 5$,
    $x - y = 0$.
11. $x^2 + 3x + y^2 = 19$,
    $2x + 7y = -17$.
12. $3xy = -42$,
    $4x + 12y = 4$.

117. Number of solutions. A quadratic equation and a linear equation have in general two and only two pairs of roots. For if the linear equation is solved for one variable, say x, and this is substituted in the quadratic equation, we get a quadratic equation determining all possible values of the other variable, in this case y, which must in general be two in number (section 105). To each one of these values of y will correspond one and only one value of x, thus affording two solutions of the pair of equations.

This result would seem at first sight to be expressed graphically by the fact that a straight line and the graph of a quadratic equation have two points of intersection. However, imaginary roots which arise in the algebraic solution of two equations do not give intersections of the graphs.

EXERCISES

1. When may, as a special case, a quadratic and a linear equation have only one solution?

2. When may a quadratic and a linear equation have imaginary solutions?

Find the real values of k for which the following equations have only one solution:

3. $x^2 + y^2 = 16,$ (1)
$x - y = k.$ (2)

Solution. $x = y + k.$

Substituting in (1), $(y + k)^2 + y^2 = 16,$

or $2y^2 + 2ky + k^2 - 16 = 0.$ (3)

As in section 106, $a = 2,\ b = 2k,\ c = k^2 - 16.$

Hence $D = b^2 - 4ac = 4k^2 - 8k^2 + 128 = -4k^2 + 128.$

For (3) to have only one solution D must equal 0.

Therefore $-4k^2 + 128 = 0,$

or $4k^2 = 128,$

and $k = \pm 4\sqrt{2}.$

4. $xy = 4,$
$x + y = k.$

5. $x^2 = ky,$
$2x + y = 8.$

6. $x + ky = 2,$
$x^2 + y^2 = 2.$

7. $y^2 = 3x,$
$3x - 3y = k.$

8. $x - y = k,$
$2x^2 - 3y = 5.$

9. $x^2 - y^2 = k,$
$2x + 3y = 7.$

10. $x^2 - y^2 = 16,$
$x + y = k.$

11. $x^2 + y^2 = 36,$
$2x - 5y = k.$

12. $4x^2 + y^2 = 9,$
$kx + y = 3.$

Graph the following equations for the given values of k:

13. $x^2 + y^2 = 16,$
$x - y = k.$
$(k = 4,\ 4\sqrt{2},\ 7.)$

Solution. For $k = 4$, $y = x - 4.$

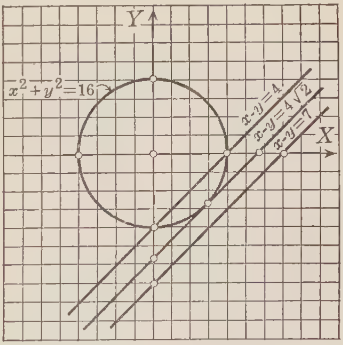
x	0	4	14
y	-4	0	10

For $k = 4\sqrt{2}$, $y = x - 4\sqrt{2}.$

x	5.66	0	3
y	0	-5.66	-2.66

For $k = 7$, $y = x - 7.$

x	0	7	3
y	-7	0	-4



$y = \pm\sqrt{16 - x^2}.$

x	4	0	-4
y	0	± 4	0

14. $xy = k,$
$x + y = 9. \quad (k = 21,\ 20\tfrac{1}{4},\ 18.)$

15. $3x^2 + y^2 = 4,$
$x - y = k. \quad (k = 2\sqrt{3},\ \tfrac{4}{3}\sqrt{3},\ \sqrt{3}.)$

16. $x^2 + y^2 = 9,$
$2x - y = k. \quad (k = 4\sqrt{5},\ 3\sqrt{5},\ 2\sqrt{5}.)$

17. $x^2 + y^2 = k,$
$x + y = 1. \quad (k = 1,\ \tfrac{1}{2},\ \tfrac{1}{3}.)$

118. Solution when neither equation is linear. In the examples previously given one equation has been linear and the other equation has been quadratic with respect to one or both variables. Often when neither of the original equations is linear a pair of equivalent (section 59) equations one or both of which are linear may sometimes be found. These latter equations may be solved by substitution.

EXERCISES

Solve the following equations:

1. $3x^2 - 2y^2 = 11, \qquad (1)$
$5x^2 + y^2 = 37. \qquad (2)$

Solution. Equations which contain only the squares of x and y may be solved by treating x^2 and y^2 as the variables, and proceeding as in the case of linear equations.

2 · (2), $\qquad 10x^2 + 2y^2 = 74. \qquad (3)$

(1) + (3), $\qquad 13x^2 = 85,$

$$x^2 = \tfrac{85}{13}, \tag{4}$$

$$x = \pm\sqrt{\tfrac{85}{13}}$$

$$= \pm\tfrac{1}{13}\sqrt{1105}.$$

Substituting (4) in (1),

$$\tfrac{255}{13} - 2y^2 = 11,$$

$$255 - 26y^2 = 143,$$

$$26y^2 = 112,$$

$$y^2 = \tfrac{112}{26} = \tfrac{56}{13},$$

$$y = \pm\sqrt{\tfrac{56}{13}}$$

$$= \pm\tfrac{1}{13}\sqrt{728}.$$

2. $x^3 + y^3 = 12,$ (1)
$x + y = 2.$ (2)

Solution. When neither of the original equations is quadratic, we can often find by division an equivalent pair of equations one of which is linear and the other quadratic, as in the following solution.

Dividing (1) by (2), $x^2 - xy + y^2 = 6.$ (3)
Squaring (2), $x^2 + 2\,xy + y^2 = 4.$
Subtracting, $-3\,xy = 2,$
$xy = -\frac{2}{3}.$ (4)

Solve (2) and (4) by substitution.

3. $x^2 + y^2 = 17,$ (1)
$xy = 4.$ (2)

Solution. When neither equation is linear, we can often obtain by addition an equation from which, by the extraction of the square root, a linear equation may be found.

$2 \cdot (2),$ $2\,xy = 8.$ (3)
(1) + (3), $x^2 + 2\,xy + y^2 = 25.$
Extracting the square root, $x + y = \pm 5.$ (4)
(1) − (3), $x^2 - 2\,xy + y^2 = 9.$
Extracting the square root, $x - y = \pm 3.$ (5)
Solving (4) and (5) as simultaneous equations,

$$x = 4,\ 1,\ -4,\ -1.$$
$$y = 1,\ 4,\ -1,\ -4.$$

Thus there are four solutions, (4, 1), (1, 4), (− 1, − 4), (− 4, − 1).

NOTE. When the sum of the exponents of the variables is the same in every term, the equation is called **homogeneous**.

Thus, $x^2 + xy = 0$ and $2\,x^2y - xy^2 - 4\,x^3 - 3\,y^3 = 0$ are homogeneous equations.

4. $x^2 - xy + 2\,y^2 = 4,$ (1)
$2\,x^2 - 3\,xy - 2\,y^2 = 6.$ (2)

Solution. When both equations are homogeneous except for a constant term, we may solve as follows:

Eliminate the constant term by multiplying (1) by 3, and (2) by 2.

$3\,x^2 - 3\,xy + 6\,y^2 = 12.$ (3)
$4\,x^2 - 6\,xy - 4\,y^2 = 12.$ (4)
(4) − (3), $x^2 - 3\,xy - 10\,y^2 = 0.$ (5)
Factoring (5), $(x - 5\,y)(x + 2\,y) = 0.$
$x = 5\,y$ or $-2\,y.$ (6)

Solve (6) and (1) by substitution.

5. $6x^2 - 7xy + 2y^2 = 0,$ (1)
$x^2 - y = 4.$ (2)

Solution. When one equation is homogeneous and the other is either linear or quadratic, we may solve them as follows:

Factoring (1), $(2x - y)(3x - 2y) = 0,$
whence $x = \frac{1}{2}y$ or $\frac{2}{3}y$.* (3)

Solve (2) and (3) by substitution.

6. $x^2 - y^2 = xy = x - y.$

7. $x^2 - y^2 = xy - 4 = x - y.$

8. $x^3 + xy^2 = 5,$
$y^3 + x^2y = 9.$

9. $x^2y = m,$
$xy^2 = n.$

10. $x(y + 1) = -4,$
$y(x + 1) = -9.$

11. $x(y - 3) = 0,$
$y(x - 3) = -6.$

12. $x^2 + y^2 = m,$
$xy = p.$

13. $xy^2 - x^2y = r,$
$xy^2 + x^2y = s.$

14. $x - xy = -1,$
$y - xy = 0.$

15. $x(x^3 - y^3) = -38,$
$y(x^3 - y^3) = -57.$

16. $4x^2 - 3y^2 = -12,$
$3x^2 - 4y^2 = -37.$

17. $2x^2 + 4y^2 = 34,$
$4x^2 - 2y^2 = 28.$

18. $x^2 + 2y^2 = 2,$
$3x^2 + 2y^2 = 2.$

19. $x^2 - xy + y^2 = 1,$
$x^2 + xy + y^2 = 3.$

HINT. Eliminate x^2 and y^2 by addition or subtraction.

20. $x^2 + 5xy + y^2 = -41,$
$x^2 - 3xy + y^2 = 79.$

21. $2x + 3y - xy = -1,$
$xy + y = 3x + 3.$

22. $x^2 + xy + y^2 = 3,$
$x^2 + y^2 = 5.$

23. $(x - y)(3 + x) = 12,$
$(x - y)(8 - y) = 14.$

24. $(x^2 + y^2)(x - y) = -25,$
$(y - x)(xy) = 12.$

25. $(x + y)^2 = 3y^2 + 6,$
$(x - y)^2 = 3x^2 - 11.$

26. $x + \sqrt[3]{xy^2} = a,$
$y + \sqrt[3]{x^2y} = b.$

27. $x\sqrt{x - y} = 3,$
$y\sqrt{x - y} = 2.$

28. $x - y = 5,$
$\sqrt{x} + \sqrt{y} = 5.$

29. $\sqrt[3]{x} - \sqrt[3]{y} = -1,$
$x - y = -7.$

---

* We observe that $y \neq 0$. For if $y = 0$ were a value that satisfies equation (1), $x = 0$ would correspond. But (0, 0) does not satisfy (2); thus $y = 0$ is not a value that can occur in the solutions of the equations.

30. $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = 13,$
$x - y = -26.$

31. $\sqrt{x} - \sqrt{y} = 0,$
$(x + y)\sqrt{xy} = 2.$

32. $x^2 - 4y^2 = 5,$
$x^2 + y^2 = \frac{10}{11}(x^2 + 2y^2).$

33. $\sqrt{x + 2} + \sqrt{y - 1} = 6,$
$x + y = 17.$

34. $5xy - 3(x - y) = 7,$
$3xy - 5(x - y) = 1.$

35. $xy + xy^{-1} = x^2 + y^2,$
$xy - xy^{-1} = 2(x^2 - y^2).$

36. $x^2 + y^2 + x + y = 12,$
$-x^2 + y^2 - x + y = 0.$

37. $x^{-2} + 2y^{-2} = \frac{11}{9},$
$x^{-2} - x^{-1}y^{-1} + y^{-2} = \frac{13}{9}.$

38. $2x^2 + 3y^2 = 125,$
$xy = 21.$

39. $x^2 + y^2 - 2(x - y) = 3,$
$x^2 + y^2 - 3(x - y) = -2.$

40. $x^2 - xy + y^2 = x + y,$
$xy = 4.$

41. $2x^2 + 3xy - 15y^2 - 26 = 0,$
$(x + 2)(y - 3) = 0.$

42. $\sqrt{x(1 + y)} + \sqrt{y(1 + x)} = 1,$
$x + y = 1.$

43. $\sqrt{2 + 3x - x^2} + \sqrt{2 + 3y - y^2} = 4,$
$x + y = 2.$

44. $\sqrt{1 - x^2}\sqrt{1 - y^2} + xy = 3,$
$x + y = 4.$

45. $\frac{1}{x} + \frac{1}{y} = \frac{1}{6},$
$x + y = -1.$

46. $\frac{x}{y} - \frac{y}{x} = \frac{8}{3},$
$2x^2 + 4y^2 = 22.$

47. $\frac{x}{y} + \frac{y}{x} = 2,$
$x^2 - y^2 = 0.$

48. $\frac{x\sqrt{x} + y\sqrt{y}}{x\sqrt{x} - y\sqrt{y}} = \frac{9}{7},$
$x^3 + y^3 = 65.$

49. $\sqrt{\frac{3x}{x - y}} + \sqrt{\frac{x - y}{3x}} = 2,$
$xy + (x - y) = 1.$

50. $\sqrt{y} - \sqrt{a + x} = \sqrt{y + x},$
$\frac{\sqrt{y + x} + \sqrt{a + x}}{\sqrt{a + x}} = \frac{5}{3}.$

51. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2,$
$\frac{bx - ay}{bx + ay} = 0.$

52. $\frac{1}{x} + \frac{1}{y} = \frac{5}{6},$
$\frac{1}{x^2} + \frac{1}{y^2} = \frac{13}{36}.$

HINT. Let $\frac{1}{x} = z$, and $\frac{1}{y} = w$.

53. $\frac{x^3}{y} + \frac{y^3}{x} = \frac{82}{3},$
$\frac{x}{y} + \frac{y}{x} = \frac{10}{3}.$

54. $x\left(1-\dfrac{x}{y}\right)=\dfrac{1}{2},$

$y\left(1-\dfrac{y}{x}\right)=-2.$

55. $\dfrac{x+1}{y+1}=\dfrac{m+1}{n+1},$

$\dfrac{x^3+1}{y^3+1}=\dfrac{m^3+1}{n^3+1}.$

56. $x=4\cdot\dfrac{y-1}{y+1},$

$y=9\cdot\dfrac{x-1}{x+1}.$

57. $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{3}{2},$

$\dfrac{1}{x+2}+\dfrac{1}{y+2}=\dfrac{7}{12}.$

58. $\dfrac{3}{x}+\dfrac{2}{y}=-1,$

$\dfrac{2}{x+3}-\dfrac{3}{y+2}=0.$

59. $\left(5-\dfrac{2x}{x+y}\right)^2+\left(5+\dfrac{2x}{x+y}\right)^2=\dfrac{458}{9},$

$xy=2.$

PROBLEMS

1. Two numbers are in the ratio 2 : 3. Their product is 96. Find the numbers.

2. What two numbers whose sum is 85 have the sum of their squares equal to 4625?

3. The sum of the squares of two numbers increased by the first number is equal to 174. When increased by the second number the sum of the squares is equal to 181. Find the two numbers.

4. The diagonal of a rectangle is 13 feet. If the two sides of the rectangle are increased 5 feet each, the area will be increased 110 square feet. Find the size of the original rectangle.

5. The diagonal of a rectangle is 90 feet. If each side were reduced 4 feet in length the diagonal would be $2\sqrt{1781}$ feet long. Find the original length of the sides.

6. The sum of two numbers is 50. The reciprocal of 2 less than the first number plus the reciprocal of 6 more than the second number is equal to $\frac{1}{12}$. Find the original numbers.

7. The sum of the squares of two numbers is 250. If the first number were increased by 1, and the second number were increased by 2, the sum of the squares would be 325. What are the numbers?

8. A highway bridge cost $5000 to construct. If the bridge had been 5 feet longer the cost would have been $6000. Find the length of the bridge and the cost per foot of length.

9. The cost per foot of a certain railway bridge is proportional to the total length of the bridge. If a bridge 100 feet long costs $100,000, find the length and cost per foot of a bridge whose total cost is $50,000.

10. A man owns a lot 200 feet wide and 300 feet long. He wishes to place a house in the center of the lot so that the house will cover one third of the area of the lot and will be surrounded by a strip of lawn of equal width on all sides. Find the dimensions of the house and the width of the lawn.

11. The front wheel of a wagon makes 198 revolutions more than the rear wheel in traveling one mile. If both wheels were the same size and the total circumference were the same as originally, the number of revolutions of each wheel in the course of a mile would have been $406\frac{2}{13}$. What is the circumference of the wheels?

12. A grocer buys eggs for $12.50. He discards as unfit for sale 2 dozen eggs and sells the remainder for a price 20 cents per dozen greater than the purchase price. If he makes a profit of $3.60 on the transaction, find the number of eggs in the original purchase.

13. A tailor buys cloth of two kinds. The first cloth costs 25 cents per yard more than the second kind, and he purchases 2 yards less of it. He pays $15 for each piece of cloth. Find the price per yard of each kind of cloth, and the amount of each which he bought.

14. A man leaves an estate of $85,000 to be divided among his 7 children. His daughters receive together $\frac{9}{17}$ of the property. Each daughter receives $5000 more than each of his sons. How many sons has he, and how much money does each of his sons and daughters receive?

15. Two farmers, A and B, purchase land. B gives $150 more for 90 acres than A does for 100 acres. A gets 1 acre more for $60 than B does. How much does each pay per acre?

16. A sum of money invested at simple interest for two years amounts at the end of that time to $1620. If the same sum were invested at the same interest rate compounded annually, the amount at the end of two years would have been $1622.40. Find the sum invested and the rate of interest.

17. A certain two-digit number divided by the sum of the digits is 3. If the digits are reversed and the resulting number is divided by the product of the digits, the result is $5\frac{1}{7}$. Find the original number.

18. What number of two digits is one greater than the sum of the squares of the digits, and 18 less than the number with the digits reversed?

19. If 1 is added to both the numerator and the denominator of a certain fraction, the resultant expression is equal to $\frac{3}{4}$. If the numerator is increased by 3, and the denominator is decreased by the same amount, the fraction becomes equal to 1. Find the original fraction.

119. Graphical meaning of homogeneous equations. Consider, for example, the homogeneous equation

$$3x^2 - 10xy - 8y^2 = 0. \qquad (1)$$

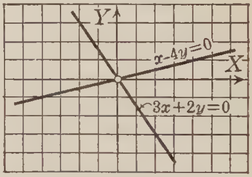
If we factor the left member of (1), we get

$$(3x + 2y)(x - 4y) = 0.$$

The roots of equation (1) are therefore

$$3x + 2y = 0, \quad \text{and} \quad x - 4y = 0.$$

These equations represent two straight lines passing through the origin, and when considered together comprise the graph of equation (1). This example may be generalized:



Any homogeneous equation of the form $ax^2 + bxy + cy^2 = 0$ *and of which the discriminant is positive is represented graphically by two straight lines through the origin.* Such an equation is equivalent to two linear equations.

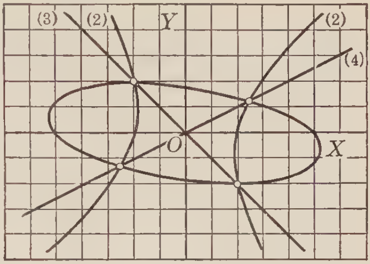
In Exercise 4, p. 147, in place of the given pair of equations, we obtained a pair of equivalent equations one of which is homogeneous and the other of which is factorable. We can learn the graphical meaning of this method of solution by studying a particular case. Consider, for example, the equations

$$x^2 + 2xy + 7y^2 = 24, \qquad (1)$$

$$2x^2 - xy - y^2 = 8. \qquad (2)$$

By eliminating the constant terms we obtain the product of the two equations $x + y = 0$ and $x - 2y = 0$. Thus the problem of solving (1) and (2) is replaced by the problem of solving the two following pairs of equations:

$$x^2 + 2xy + 7y^2 = 24,$$
$$x + y = 0; \quad (3)$$
$$x^2 + 2xy + 7y^2 = 24,$$
$$x - 2y = 0. \quad (4)$$



The graphical meaning of this method of solving the equations (1) and (2) is seen in the fact that the problem of finding the points of intersection of the graph of equation (1) with that of (2) is changed to that of finding the intersection of the graph of (1) with a pair of straight lines, (3) and (4). This appears from the figure, where the lines and curves are numbered to correspond with the equations. The closed curve is the graph of (1).

CHAPTER XII

MATHEMATICAL INDUCTION

120. General statement. Many formulas are capable of simple verification in special cases, while for the general case a direct proof is difficult.

If we ask whether $(x^n - 1)$ is divisible by $(x - 1)$, it is easy to make the actual division for any particular value of n, as $n = 2$ or $n = 3$. But if $(x^3 - 1)$ is shown divisible by $(x - 1)$, we are no wiser than before concerning the divisibility of $(x^4 - 1)$. Suppose, however, we can prove that the divisibility for $n = (k + 1)$ is always possible provided the divisibility for $n = k$ is possible, no matter what value k may happen to have. Then, since we already know that $(x^2 - 1)$ and $(x^3 - 1)$ are divisible by $(x - 1)$, we are made certain that divisibility is possible in the case of $n = 4$ and $n = 5$ and so on, no matter how large a value n may assume.

Let us assume that $(x^k - 1)$ is divisible by $(x - 1)$. This assumption is perfectly valid provided k has a numeric value for which we have verified the possibility of the division, such as $k = 2$ or 3. For we know that $x^3 - 1 = (x - 1)(x^2 + x + 1)$.

Now $x^{k+1} - 1 \equiv x^{k+1} - x^k + x^k - 1 \equiv x^k(x - 1) + x^k - 1.$

But $(x - 1)$ is obviously a factor of the right-hand member of this equation and consequently is a factor of $(x^{k+1} - 1)$. Thus the divisibility of $(x^k - 1)$ by $(x - 1)$ is established for any integral value of n greater than the one for which the division has actually been carried out.

For the complete proof of a formula by mathematical induction we must make the two distinct steps as shown on the following page.

I. Establish the theorem for some particular case or cases, preferably for $n = 1$ and $n = 2$.

II. Show that the theorem for $n = (k + 1)$ follows from its assumed validity for $n = k$.

EXAMPLE

Prove that the sum of the integers from 1 to n is

$$\frac{n(n+1)}{2}.$$

Proof. Restating the theorem in algebraic symbols, we have

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

I. This theorem is true for $n = 1$,

since $$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1.$$

The theorem is also true for $n = 2$,

since $$1 + 2 = \frac{2(2+1)}{2} = 3.$$

II. Assuming the theorem for $n = k$,*

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}. \tag{1}$$

Adding $(k + 1)$ to both sides of the equation,

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2}, \end{aligned}$$

which is the form desired, that is, $(k + 1)$ replaces k in the formula (1).

EXERCISES

Prove by mathematical induction that

1. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.
2. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$.
3. $3 + 6 + 9 + \cdots + 3n = \frac{3}{2}(n + 1)n$.

---

* It should be remembered that this statement is true only for such values of k as have been verified specifically, that is, 1 and 2.

4. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}\, n(n+1)(2n+1)$.

5. $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$.

6. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \{\frac{1}{2}[n(n+1)]\}^2$.

7. $4^2 + 7^2 + 10^2 + \cdots + (3n+1)^2 = \frac{1}{2}\, n(6n^2 + 15n + 11)$.

8. $x^n - y^n$ is divisible by $x - y$ for any integral values of n.

9. $x^{2n} - y^{2n}$ is divisible by $x + y$ for any integral values of n.

10. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{1}{3}\, n(n+1)(n+2)$.

11. $1 \cdot 1 + 2 \cdot 3^2 + 3 \cdot 5^2 + \cdots + n(2n-1)^2$
$= \frac{1}{6}\, n(n+1)(6n^2 - 2n - 1)$.

12. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n+1)(n+2)$
$= \frac{1}{4}\, n(n+1)(n+2)(n+3)$.

13. $(1^3 + 2^3 + 3^3 + \cdots + n^3) + 3(1^5 + 2^5 + 3^5 + \cdots + n^5)$
$= 4(1 + 2 + 3 + \cdots + n)^3$.

14. $1 \cdot 3 + 3 \cdot 3^2 + 5 \cdot 3^3 + \cdots + (2n-1)3^n = 3^{n+1}(n-1) + 3$.

15. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n+1)(2n-1)}{3}$.

16. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

17. $2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 + \cdots + (n+1)(n+4) = \dfrac{n}{3}(n+4)(n+5)$.

18. $2 \cdot 4 + 4 \cdot 6 + 6 \cdot 8 + \cdots + 2n(2n+2)$
$= \dfrac{n}{3}(2n+2)(2n+4)$.

19. A pyramid of shot stands on a triangular base having t shot on a side. How many shot are there in the pile?

20. A pile of shot is arranged in a pyramid having a square base so that there are k shot on a side of the base. Find the number of shot in the pile.

CHAPTER XIII

BINOMIAL THEOREM

121. Statement of the binomial theorem. When in previous problems any power of a binomial of the form $(a+b)$ has been required, we have obtained the result by direct multiplication. We can, however, derive a general law known as the **binomial theorem**, which gives the form of development of $(a+b)^n$, where n is any positive integer and a and b are any algebraic or arithmetic expressions. This law is as follows:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + b^n.$$

From an observation of this expression we obtain the following

Rule for the development of $(a+b)^n$.

The first term is a^n.

The second term is $na^{n-1}b$.

The sum of the exponents of a and b in each term is n.

The exponent of a in each term is 1 less than that of the preceding term.

The exponent of b in each term is 1 greater than it is in the preceding term.

If in any term the coefficient is multiplied by the exponent of a, and this product is divided by the exponent of b increased by 1, the result is the coefficient of the next following term.

EXERCISES

1. What is the first term of the development of $(a+b)^n$ when $n=2$? $n=3$? $n=4$?

2. What is the second term of the development of $(a+b)^n$ when $n=2$? $n=3$? $n=4$?

3. Apply the preceding rule to find the third term of the development of $(a+b)^n$ when $n=2$; $n=3$; $n=4$.

4. Apply the preceding rule to find the fourth term of $(a+b)^n$ when $n=2$; $n=3$; $n=4$.

NOTE. In practice it is often more convenient first to write down all the terms with their proper exponents, and then to form the successive coefficients.

122. The general term. A study of the rule in section 121 shows us that the third term of the binomial expansion is

$$\frac{n(n-1)}{2}a^{n-2}b^2,$$

and that the fourth term is

$$\frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3.$$

In a precisely similar manner we may write the expression for the rth term of the expansion,

$$\frac{n(n-1)(n-2)\cdots(n-r+2)}{2\cdot 3\cdot 4\cdots(r-1)}a^{n-r+1}b^{r-1}.$$

It will sometimes be more convenient to use the $(r+1)$th term instead of the rth term, as the expression is somewhat simpler in form,

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{2\cdot 3\cdot 4\cdots r}a^{n-r}b^r.$$

EXERCISES

1. Find the third term of $(a+b)^5$.

2. Find the seventh term of $(a+b)^{12}$.

3. Find the fifth term of $(a+b)^4$.

4. What is the tenth term of $(a+b)^8$? the eleventh term?

5. Does the binomial theorem hold true for the expansion of $(a+b)^n$ when $n=1$?

6. What is the fifteenth term of $(a+b)^{15}$?

7. What is the fourth term of $(a-b)^6$?

8. What is the fifth term of $(a-b)^6$?

9. Find the third term from the end of $(a+b)^7$.

10. Find the fourth term from the end of $(a-b)^8$.

123. Proof of the binomial theorem. In sections 121 and 122 the following form for the binomial expansion was obtained. The validity of the development when $n = 2$ and $n = 3$ has been shown, and the stage is set for a proof of the formula in general by means of mathematical induction.

$$\begin{aligned}(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2}a^{k-2}b^2 + \cdots \\ + \frac{k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)}a^{k-r+1}b^{r-1} \\ + \frac{k(k-1)\cdots(k-r+1)}{1\cdot 2\cdots r}a^{k-r}b^r + \cdots + kab^{k-1} + b^k.\end{aligned}$$

Multiply both members of this equation by $(a + b)$, retaining only the first two terms, the last term, and the $(r + 1)$th term.

$$\begin{array}{l} (a+b)^{k+1} = \\ a^{k+1} + \qquad ka^kb + \cdots + \dfrac{k(k-1)\cdots(k-r+1)}{1\cdot 2\cdots r}a^{k-r+1}b^r + \cdots \\ \qquad\qquad\qquad a^kb + \cdots + \dfrac{k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)}a^{k-r+1}b^r + \cdots + b^{k+1} \\ \hline a^{k+1} + (k+1)a^kb + \cdots + \dfrac{(k+1)k\cdots(k-r+2)}{1\cdot 2\cdots r}a^{k-r+1}b^r + \cdots + b^{k+1} \end{array}$$

The derivation of the middle, or $(r + 1)$th, term in the complete expression which we have just obtained is somewhat involved. A more complete exposition of the steps required for the simplification of this term follows:

Given $$\frac{k(k-1)\cdots(k-r+1)}{1\cdot 2\cdots r} + \frac{k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)}.$$

Combining, we have

$$\frac{[k(k-1)\cdots(k-r+1)] + r[k(k-1)\cdots(k-r+2)]}{1\cdot 2\cdots r}.$$

Factoring, we transform the expression to

$$\frac{[k(k-1)\cdots(k-r+2)][(k-r+1)+r]}{1\cdot 2\cdots r}.$$

Rearranging and simplifying, we reduce it to

$$\frac{(k+1)k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots r}.$$

This is the coefficient of the $(r + 1)$th term.

It is immediately observed that the first two terms and the last one are in the form required by the rule of section 121, where $n = k + 1$. If in the formula for the $(r+1)$th term, n be replaced by $(k+1)$, we obtain the $(r+1)$th term in the right-hand member above. That is, replacing k by $(k+1)$ in

$$\frac{k(k-1)\cdots(k-r+1)}{1\cdot 2\cdots r}\,a^{k-r}b^r,$$

we get

$$\frac{(k+1)k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots r}\,a^{k-r+1}b^r.$$

Hence the rule of section 121, which we have verified for small values of n, holds for any positive integral value of n.

EXERCISES

Expand by means of the binomial theorem:

1. $\left(\frac{\sqrt{x}}{y^2} - \frac{y^3}{\sqrt{x}}\right)^8$.

Solution. $$\frac{x^4}{y^{16}} + 8\left(\frac{\sqrt{x}}{y^2}\right)^7\left(-\frac{y^3}{\sqrt{x}}\right) + \frac{8\cdot 7}{2}\left(\frac{\sqrt{x}}{y^2}\right)^6\left(-\frac{y^3}{\sqrt{x}}\right)^2$$
$$+\frac{8\cdot 7\cdot 6}{2\cdot 3}\left(\frac{\sqrt{x}}{y^2}\right)^5\left(-\frac{y^3}{\sqrt{x}}\right)^3 + \frac{8\cdot 7\cdot 6\cdot 5}{2\cdot 3\cdot 4}\left(\frac{\sqrt{x}}{y^2}\right)^4\left(-\frac{y^3}{\sqrt{x}}\right)^4$$
$$+\frac{8\cdot 7\cdot 6\cdot 5\cdot 4}{2\cdot 3\cdot 4\cdot 5}\left(\frac{\sqrt{x}}{y^2}\right)^3\left(-\frac{y^3}{\sqrt{x}}\right)^5 + \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{2\cdot 3\cdot 4\cdot 5\cdot 6}\left(\frac{\sqrt{x}}{y^2}\right)^2\left(-\frac{y^3}{\sqrt{x}}\right)^6$$
$$+\frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\left(\frac{\sqrt{x}}{y^2}\right)\left(-\frac{y^3}{\sqrt{x}}\right)^7 + \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}\left(-\frac{y^3}{\sqrt{x}}\right)^8$$
$$= \frac{x^4}{y^{16}} - 8\,\frac{x^3}{y^{11}} + 28\,\frac{x^2}{y^6} - 56\,\frac{x}{y} + 70\,y^4 - 56\,\frac{y^9}{x} + 28\,\frac{y^{14}}{x^2} - 8\,\frac{y^{19}}{x^3} + \frac{y^{24}}{x^4}.$$

2. $\left(\frac{1}{2} + \frac{2}{3}a\right)^5$.

3. $(\sqrt{x} + \sqrt[3]{y})^4$.

4. $\left(\frac{1}{y} + y\right)^7$.

5. $\left(\frac{x}{y} - \frac{y}{x}\right)^5$.

6. $\left(\frac{3a}{2b} + \frac{2a}{3b}\right)^6$.

7. $\left(\frac{x^2}{4} - \frac{3}{x}\right)^6$.

8. $\left(\frac{\sqrt{m}}{5} - \frac{2\sqrt{p}}{3}\right)^4$.

9. $\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{3}\right)^5$.

10. $(1 - \sqrt{a})^4 - (1 + \sqrt{a})^4$.

11. $(\sqrt{x} - \sqrt{y})^5 + (\sqrt{x} + \sqrt{y})^5$.

12. Find the fifth term of $(3a - 2b)^7$.

13. Find the eighth term of $\left(a + \frac{1}{a}\right)^{10}$.

14. Find the fourth term of $\left(\frac{a}{3b} + \frac{3b}{a}\right)^{12}$.

15. Find the thirteenth term of $\left(\frac{\sqrt{a}}{x} - \frac{\sqrt{x}}{a}\right)^{20}$.

16. Find the tenth term of $\left(3x\sqrt{y} + \frac{1}{3y\sqrt{x}}\right)^{15}$.

17. Find the ninth term of $\left(\frac{\sqrt[3]{x}}{\sqrt{y}} - \frac{\sqrt[3]{y}}{\sqrt{x}}\right)^{11}$.

18. Find the value of $(.8)^5$, correct to three decimal places.

Solution. $(.8)^5 = (1 - .2)^5$

$$\begin{aligned} &= 1^5 - 5(1)^4(.2) + \frac{5 \cdot 4}{2}(1)^3(.2)^2 - \frac{5 \cdot 4 \cdot 3}{2 \cdot 3}(1)^2(.2)^3 \\ &\quad + \frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 4}(1)(.2)^4 - \cdots \\ &= 1 - 1 + 10(.04) - 10(.008) + 5(.0016) - \cdots \\ &= 1 - 1 + .4 - .08 + .008 - \cdots \\ &= .328. \end{aligned}$$

Further development of this expression would give no greater accuracy in the third decimal place.

Find the value of the following expressions, correct to three significant figures:

19. $(1.1)^7$.

20. $(.9)^9$.

21. $(.99)^{10}$.

22. $(1.02)^6$.

23. $(\frac{24}{25})^8$.

24. $(\frac{51}{50})^{12}$.

25. $(1.03)^{15}$.

26. $(9.9)^7$.

27. In what term of the expansion of $(a + b)^{18}$ does a^{10} occur?

28. What must be the characteristic of n in $(a + b)^n$ so that a and b will have equal exponents in one term of the expansion?

29. What must be true of n in $(a + b)^n$ so that there will be an even number of terms in the expansion?

30. Give the first three and the last two terms in the development of $\left(\sqrt[4]{x^2y^3} - \frac{3}{x^3y^{-\frac{3}{4}}}\right)^{13}$.

31. What will $300 invested at 4 % interest amount to in 5 years, if the interest is compounded annually?

HINT. An examination of the formulas in section 95 will show that this exercise involves merely a binomial expansion. The terms of the expansion may be simplified conveniently by the use of logarithms.

32. How much will $1 amount to in 6 years at 6 % interest compounded semiannually?

33. What will $1250 amount to in 3 years if it is invested at $3\frac{1}{2}$ % interest compounded quarterly?

34. What will $50 amount to in 1 year if it is invested at 3 % interest compounded monthly?

CHAPTER XIV

ARITHMETIC PROGRESSION

124. Definitions. A series of numbers such that each number in the series is obtained by adding a fixed number to the preceding one is called an **arithmetic progression.**

The fixed number which is added to one term of the progression to obtain the next following term is called the **common difference.**

It should be noted that the common difference may be either a positive or a negative number. Hence the progression may be either an increasing or a decreasing series of numbers.

Thus $2, 4, 6, 8, \cdots$ is an arithmetic progression with the common difference 2. The set of numbers $12, 6, 0, -6, -12, \cdots$ is also an arithmetic progression with the common difference -6.

EXERCISES

Determine whether the following series are in arithmetic progression. If so, find the common difference.

1. $7, 6\frac{1}{3}, 5\frac{2}{3}, \cdots$.
2. $6, 5\frac{1}{2}, 4, \cdots$.
3. $12, 10\frac{5}{6}, 9\frac{2}{3}, \cdots$.
4. $3, 1, -1, \cdots$.
5. $\sqrt{\frac{1}{3}}, 2\sqrt{\frac{1}{3}}, \sqrt{3}, \cdots$.
6. $-10, -7\frac{1}{3}, -4\frac{2}{3}, \cdots$.
7. $\dfrac{3}{\sqrt{3}}, \dfrac{5}{\sqrt{3}}, \dfrac{8}{\sqrt{3}}, \cdots$.
8. $\dfrac{\sqrt{3}+1}{3}, \dfrac{\sqrt{3}}{3}, \dfrac{1}{3\sqrt{3}+1}, \cdots$.
9. $4, 1\frac{1}{2}, -1, \cdots$.
10. $\dfrac{\sqrt{2}}{4}, \frac{1}{2}(\sqrt{2}-1), \dfrac{13}{4(\sqrt{2}+1)}, \cdots$.

125. The *n*th term. The terms of an arithmetic progression in which a is the first term and d is the common difference are as follows:

$$a, a+d, a+2d, a+3d, \cdots.$$

The coefficient of d is seen to be 1 in the second term, 2 in the third term, and in general always one less than the number of the term in which it appears. If we call l the nth term, we have

$$l = a + (n - 1)d.$$

We may also write the series in which l is the nth term as follows:

$$a,\ a + d,\ a + 2\,d,\ \cdots,\ l - 2\,d,\ l - d,\ l.$$

126. The sum of the series. We may obtain a formula for computing the sum of the first n terms of an arithmetic progression by the following

Theorem. *The sum s of the first n terms of the arithmetic progression $a,\ a + d,\ a + 2\,d,\ \cdots,\ l - d,\ l$ is*

$$s = \frac{n}{2}(a + l).$$

By definition,

$$s = a + (a + d) + (a + 2\,d) + \cdots + (l - 2\,d) + (l - d) + l. \quad (1)$$

Inverting the order of the terms of the right-hand member of (1),

$$s = l + (l - d) + (l - 2\,d) + \cdots + (a + 2\,d) + (a + d) + a. \quad (2)$$

Adding (1) and (2) term by term,

$$2\,s = (l + a) + (l + a) + (l + a) + \cdots + (l + a) + (l + a) + (l + a)$$
$$= n(l + a).$$

Thus

$$s = \frac{n}{2}(a + l).$$

127. Arithmetic means. The terms of an arithmetic progression between a given term and a subsequent term are called **arithmetic means** between those terms. By *the* arithmetic mean of two numbers is meant the number which is the second term of an arithmetic series in which the given numbers are considered the first and third terms. Thus the arithmetic mean of two numbers a and b is $\frac{a + b}{2}$, since the numbers a, $\frac{a + b}{2}$, b are in arithmetic progression, with the common difference $\frac{b - a}{2}$.

The two formulas $$l = a + (n-1)d, \qquad \text{(I)}$$
$$s = \frac{n}{2}(a + l) \qquad \text{(II)}$$
contain the **elements** a, l, s, n, d. Evidently when any three are known, the remaining two may be found by solving the equations (I) and (II) simultaneously.

EXERCISES

1. Find the tenth term and the sum of the series 8, 6, 4, · · ·.

Solution. $n = 10,\ a = 8,\ d = -2.$
$$l = a + (n-1)d = 8 + (9)(-2) = -10.$$
$$s = \tfrac{1}{2}n(a + l) = 5(8 - 10) = -10.$$

2. $l = 5,\ n = 2,\ d = -1$. Find a and s.

3. $d = 5,\ s = 30,\ a = 0$. Find l and n.

4. $d = -4,\ a = 7,\ n = 12$. Find l and s.

5. $d = \frac{1}{4},\ l = \frac{1}{2},\ s = \frac{1}{2}$. Find a and n.

6. $a = 0,\ s = -6,\ l = -3$. Find n and d.

7. $s = 100,\ l = 19,\ n = 10$. Find a and d.

8. $l = 99,\ n = 99,\ a = 1$. Find s and d.

9. $s = 435,\ n = 15,\ a = 1$. Find l and d.

10. $s = 1\frac{1}{2},\ n = 5,\ d = 1$. Find a and l.

11. $d = \frac{1}{6},\ l = 1,\ a = \frac{1}{3}$. Find n and s.

12. Find expressions for a and n in terms of d, l, and s.

13. Find expressions for d and s in terms of a, n, and l.

14. Find expressions for l and n in terms of a, d, and s.

15. Find expressions for a and s in terms of d, l, and n.

16. Find expressions for d and n in terms of a, l, and s.

17. Insert 4 arithmetic means between 2 and 12.

HINT. $a = 2,\ l = 12,\ n = 6$. Find d.

18. Insert 5 arithmetic means between 3 and 45.

19. Insert 3 arithmetic means between 12 and -8.

20. Insert 2 arithmetic means between $\dfrac{\sqrt{3}}{2}$ and $\dfrac{\sqrt{3}+1}{3}$.

21. Insert 3 arithmetic means between $\dfrac{\sqrt{2}-1}{3}$ and $\dfrac{\sqrt{2}+1}{3}$.

22. Insert $(n-2)$ arithmetic means between a and l.

NOTE. An exercise may often be solved more simply if, instead of assuming the series $x, x+y, x+2y, \cdots$, we assume $x-y, x, x+y$ when three terms are required; or $x-2y, x-y, x, x+y, x+2y$ when five terms are required, etc.

23. Find the fifth term and the sum of the series $3, 0, -3, \cdots$.

24. Find the tenth term and the sum of the series $4, 7, 10, \cdots$.

25. Find the sixth term and the sum of the series $5\frac{1}{3}, 4\frac{1}{6}, 3, \cdots$.

26. Find the sixth term and the sum of the series $1, -5, -11, \cdots$.

27. Find the twelfth term and the sum of the series $101, 98, 95, \cdots$.

28. Find the twenty-first term and the sum of the series $13, 15\frac{1}{2}, 18, \cdots$.

29. Find the fifth term and the sum of the series

$$\frac{\sqrt{3}}{2}, \frac{1}{2}\sqrt{3}+\frac{1}{3}\sqrt{2}, \frac{3\sqrt{3}+4\sqrt{2}}{6}, \cdots.$$

30. Find the eighth term and the sum of the series

$$\frac{3}{\sqrt{2}}, \frac{16}{5\sqrt{2}}, \frac{17\sqrt{2}}{10}, \cdots.$$

31. Find the twelfth term and the sum of the series

$$\frac{\sqrt{3}+1}{5}, \frac{2\sqrt{3}}{5}, \frac{3\sqrt{3}-1}{5}, \cdots.$$

32. Find the seventh term and the sum of the series $2x, 5x, 8x, \cdots$.

33. Find the eleventh term and the sum of the series $a-2b, a-b, a, \cdots$.

34. The sum of the first 3 terms of an arithmetic progression is 6. The sum of the first 10 terms is 55. Find the series.

35. The sum of the first 7 terms of an arithmetic progression is 7, and the sum of the first 12 terms is 72. What are the first 3 terms of the series?

36. The sum of the first 4 terms of an arithmetic progression is 16, and the sum of the squares of the same 4 terms is 84. Find the progression.

37. The velocity of a falling body at the end of each second since it started to fall may be considered as a term in an arithmetic progression. If the velocity at the end of the fifth second is 160 feet per second, and the body started with no initial velocity, find the increase in velocity per second. (This increase in velocity is called the *acceleration.*)

HINT. Velocity at the end of the first second is the second term of the progression. The acceleration is the common difference.

38. If a body falls 89 feet per second at the end of the third second and 185 feet per second at the end of the sixth second, determine its initial velocity and its acceleration.

39. A body moves upward with a velocity of 100 feet per second at the end of the fifth second, and downward with a velocity of 20 feet per second at the end of the eighth second. Find the initial velocity and the acceleration.

HINT. If downward velocity is considered positive, upward velocity must be negative.

40. A body is moving downward 213 feet per second at the end of the fourth second, and downward at a speed of 63 feet per second at the end of the second second. What is the acceleration and what was the initial velocity?

41. A body has an initial velocity of 35 feet per second. At the end of 3 seconds its velocity is 50 feet per second in the same direction. What will be its velocity at the end of 15 seconds if the acceleration is constant? What is the acceleration?

42. Two arithmetic progressions have the same first terms. The common differences are in the ratio of 2 : 3. Find a common difference and a first term for each series such that the third term of one series will be 13, while the fifth term of the other will be 29.

43. The four digits of a number are in arithmetic progression. The sum of the digits is 24, and the last digit is 3 times the first digit. Find the number.

CHAPTER XV

GEOMETRIC PROGRESSION

128. Definitions. A series of numbers such that each number in the series is obtained by multiplying the number immediately preceding by a constant factor is called a **geometric progression.**

The constant factor by which each number of the series is multiplied to give the following number is called the **ratio.**

It should be remembered that the ratio, like the common difference in arithmetic progressions, may be either a positive or a negative number. Likewise it may be either an integer or a fraction.

The geometric series 1, 2, 4, 8, $\cdots$ has the ratio 2. The geometric progression $\sqrt{3}$, 3, $3\sqrt{3}$, 9, $\cdots$ has the ratio $\sqrt{3}$.

EXERCISES

Determine which of the following series are in geometric progression, and find the ratio:

1. $6, 2, \frac{2}{3}, \cdots.$

2. $7, 14, 28, \cdots.$

3. $12, -2, \frac{1}{3}, \cdots.$

4. $4, -2, -1, \cdots.$

5. $\frac{1}{2}\sqrt{2}, \sqrt{2}, \sqrt{8}, \cdots.$

6. $\dfrac{2}{\sqrt{3}}, \dfrac{2}{3}, \dfrac{2\sqrt{3}}{9}, \cdots.$

7. $-\dfrac{3}{\sqrt{2}}, \dfrac{3}{2}, -\dfrac{3}{2\sqrt{2}}, \cdots.$

8. $\dfrac{1}{\sqrt{5}}, \dfrac{1}{5\sqrt{2}}, \dfrac{1}{10\sqrt{5}}, \cdots.$

9. $\sqrt{\frac{7}{8}}, \frac{1}{4}\sqrt{7}, \frac{1}{8}\sqrt{14}, \cdots.$

10. $\dfrac{\sqrt{5}}{7}, \sqrt{\frac{5}{7}}, \sqrt{5}, \cdots.$

11. $\dfrac{1}{\sqrt{2}+\sqrt{5}}, -\frac{1}{3}(\sqrt{2}-\sqrt{5}), -\dfrac{7}{3}+2\sqrt{\frac{7}{9}}, \cdots.$

12. $\sqrt{3}+1, 2, 2\sqrt{3}-2, \cdots.$

129. The nth term. The terms of a geometric progression in which a is the first term and r the ratio are as follows:

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots.$$

The power of r in the second term is 1, in the third term is 2, and in fact is always one less than the number of the term. If we call l the nth term, we have the following expression for the nth term:

$$l = ar^{n-1}.$$

130. The sum of the series. We obtain a formula for finding the sum of the first n terms of a geometric progression by the following

Theorem. The sum s of the first n terms of the geometric progression $a, ar, ar^2, \cdots$ is

$$s = \frac{a - rl}{1 - r}.$$

By definition,
$$\begin{aligned} s &= a + ar + ar^2 + \cdots + ar^{n-1} \\ &= a(1 + r + r^2 + \cdots + r^{n-1}) \\ &= a\left(\frac{1 - r^n}{1 - r}\right) \\ &= \frac{a - rar^{n-1}}{1 - r} = \frac{a - rl}{1 - r}. \end{aligned}$$

131. Geometric means. The $(n-2)$ terms between the first and nth term of a geometric progression are called the **geometric means** between those terms.

If one geometric mean is inserted between two numbers, it is called *the* geometric mean of those numbers. Thus the geometric mean between a and b is $\sqrt{ab}$.

The two fundamental formulas

$$l = ar^{n-1} \tag{I}$$

and

$$s = \frac{a(1 - r^n)}{1 - r} = \frac{a - rl}{1 - r} \tag{II}$$

contain the five **elements** a, l, r, n, s, any two of which may be found if the remaining three are given.

EXERCISES

1. Find the tenth term and the sum of the progression 2, 6, 18, $\cdots$.

Solution. $a = 2$, $r = 3$, $n = 10$.

From (I), $l = 2 \cdot 3^9 = 39{,}366.$

From (II), $s = \dfrac{2 - 3 \cdot 39366}{1 - 3} = 59{,}048.$

2. Find the fifth term and the sum of the progression 3, 1, $\frac{1}{3}$, $\cdots$.

3. Find the eighth term and the sum of the progression 3, 9, 27, $\cdots$.

4. Find the seventh term and the sum of the progression 1, 2, 4, $\cdots$.

5. Find the twelfth term and the sum of the progression 8, 4, 2, $\cdots$.

6. Find the ninth term and the sum of the progression $\sqrt{2}$, 2, $2\sqrt{2}$, $\cdots$.

7. Find the eleventh term and the sum of the progression $\sqrt[3]{9}$, $3\sqrt[3]{3}$, 9, $\cdots$.

8. Find the sixth term and the sum of the progression $\frac{1}{98}$, $\frac{1}{14}$, $\frac{1}{2}$, $\cdots$.

9. Find the tenth term and the sum of the progression $\sqrt{2} + 1$, 1, $\sqrt{2} - 1$, $\cdots$.

10. Find the fourth term and the sum of the progression $\frac{1}{7}$, $\frac{3 + \sqrt{2}}{7}$, $\frac{11 + 6\sqrt{2}}{7}$, $\cdots$.

11. Find the sixth term and the sum of the progression $\frac{1}{2}$, $\frac{1}{2}\sqrt{2}$, 1, $\cdots$.

12. Find the fifth term and the sum of the progression $\sqrt{\frac{1}{3}}$, 1, $\sqrt{3}$, $\cdots$.

13. Find the seventh term and the sum of the progression $\sqrt{5}$, 1, $\frac{1}{5}\sqrt{5}$, $\cdots$.

14. Find the eighth term and the sum of the progression $\sqrt{\frac{2}{3}}$, $\frac{2\sqrt{3}}{3}$, $\frac{2\sqrt{6}}{3}$, $\cdots$.

15. Find the fifth term and the sum of the progression $\frac{\sqrt{5}}{3}$, $\sqrt{\frac{5}{3}}$, $\sqrt{5}$, $\cdots$.

16. Insert 2 geometric means between 3 and 81.

17. Insert 3 geometric means between 1 and 16.

18. Insert 6 geometric means between $\frac{1}{2}$ and -64.

19. Insert 4 geometric means between 1 and $\frac{1}{32}$.

20. Insert 5 geometric means between 1 and 2.

21. Insert 1 geometric mean between 3 and 4.

22. Insert 3 geometric means between 7 and 13.

23. Insert 5 geometric means between 9 and $\frac{1}{81}$.

24. Insert 2 geometric means between $\sqrt{2}$ and $\sqrt{5}$.

25. Find the geometric mean between 13 and 18.

26. Insert 2 geometric means between 2 and $\frac{1}{8}$.

27. Find the geometric mean between 7 and 35.

28. Find the geometric mean between $\sqrt{x}$ and $\sqrt{y}$.

29. Insert 2 geometric means between $\frac{1}{\sqrt{2}}$ and 2.

30. Find the geometric mean between $\frac{a+b}{a}$ and $\frac{a+b}{b}$.

31. The first and third terms of a geometric progression are 4 and 64. What is the progression?

32. The third and seventh terms of a geometric progression are 25 and $\frac{1}{25}$. Find the progression.

33. The geometric mean of two numbers is 4, and their difference is 6. Find the numbers.

34. The second term of a geometric progression is 18, and the sixth term is 144. Find the first term and the ratio of the series.

35. If the same number be added to each term of a geometric progression, is the result a geometric progression?

36. If each term of a geometric progression be multiplied by the same constant, is the result a geometric progression? If so, what is the ratio?

37. The sum of 3 numbers in geometric progression is 14, and the product of the first and second is 32. Find the numbers.

38. The sum of 4 numbers in geometric progression is 240. The sum of the first and fourth is 168. Find the numbers.

39. If a man works for 1 cent the first day, 2 cents the second day, and 4 cents the third day, how much will he receive on the thirtieth day? How much will he have received in all up to that time?

40. A ball falling from a height of 4 feet rebounds $\frac{1}{3}$ of the previous fall each time it strikes the ground. How far has it traveled at the time it strikes the ground the tenth time? How high will its next bounce be?

41. A sound echoes across a valley in such a way that each echo is $\frac{2}{3}$ as loud as the preceding one. What is the intensity of the fifth echo as compared with the original sound?

42. Two numbers differ by 48. The geometric mean is 8 less than the arithmetic mean. Find the numbers.

43. A sum of money invested at compound interest doubles itself every 13 years. What will \$1 amount to at the end of 65 years?

44. A man owes \$50. If each week he pays half the amount still due, how long before he will owe less than \$1?

45. The sum of the first and last of 4 numbers in geometric progression is 630. The sum of the second and third numbers is 150. Find the numbers.

132. Infinite series. When the number of terms of a geometric progression is unlimited it is called an **infinite geometric series.**

In the series $a, ar, ar^2, \cdots$, when $r > 1$, evidently each term is larger than the preceding term. The series is then called increasing. When $r < 1$, each term is smaller than the preceding term, and the series is called decreasing.

Now in any case,

$$s = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

When $r > 1$, evidently r^n becomes very large for large values of n. For this case, then, the sum of the first n terms becomes very large for large values of n. In fact, we can take enough terms so that s will exceed any number we may choose. If, however, $r < 1$, as n increases in value r^n becomes smaller and smaller. In fact, we can choose n large enough to make r^n as small as we may wish, or, as we say, approach 0 as a limit. But since r^n may be made as small as we wish, ar^n also approaches 0 as a limit, and consequently $\frac{ar^n}{1-r}$ approaches 0 as a limit. But for any value of n, $\frac{a}{1-r}$ remains unchanged. Thus when $r < 1$ the value of the sum of the first n terms approaches $\frac{a}{1-r}$ as n becomes very great. This we express in other words by asserting that when $r < 1$ the sum of the infinite series

$$a + ar + ar^2 + \cdots$$

is

$$s_\infty = \frac{a}{1-r}.$$

EXERCISES

Find the sum of the following infinite series:

1. $8 + 4 + 2 + \cdots$.

Solution. $a = 8$, $r = \frac{1}{2}$.

$$s_\infty = \frac{a}{1-r} = \frac{8}{\frac{1}{2}} = 16.$$

2. $3 + 1\frac{1}{2} + \frac{3}{4} + \cdots$.

3. $18 + 6 + 2 + \cdots$.

4. $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \cdots$.

5. $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$.

6. $\frac{7}{8} + \frac{1}{4} + \frac{1}{14} + \cdots$.

7. $3 + 1 + \frac{1}{3} + \cdots$.

8. $7 + 3.5 + 1.75 + \cdots$.

9. $\sqrt{5} + 1 + \frac{\sqrt{5}}{5} + \cdots$.

10. $(\sqrt{3} + 1) + 2 + 2(\sqrt{3} - 1) + \cdots$.

Find n so that the sum of n terms shall differ from s_∞ by less than .001.

11. $12 + 4 + 1\frac{1}{3} + \cdots$.

Solution. $a = 12$, $r = \frac{1}{3}$.

$$s_n = \frac{a}{1-r} - \frac{ar^n}{1-r} = s_\infty - \frac{ar^n}{1-r}.$$

$$s_\infty - s_n = \frac{ar^n}{1-r}.$$

We must find n so that $\frac{ar^n}{1-r}$ is less than .001.

$$\frac{12\cdot(\frac{1}{3})^n}{1-\frac{1}{3}}=\frac{12\cdot 3}{2\cdot 3^n}=\frac{18}{3^n}=\frac{2}{3^{n-2}}.$$

By trial we see that $\frac{2}{3^{n-2}}$ is less than .001 when $n-2=7$, or $n=9$, since $\frac{2}{3^{9-2}}=\frac{2}{2187}$.

12. $81+9+1+\cdots$.

13. $1+\frac{1}{3}+\frac{1}{9}+\cdots$.

14. $90+9+.9+\cdots$.

15. $21+7+2\frac{1}{3}+\cdots$.

16. $16+2+\frac{1}{4}+\cdots$.

17. $25+5+1+\cdots$.

Find the value of the following recurring decimal fractions:

18. $.333\cdots$.

Solution. This expression may be written in the form

$$\tfrac{3}{10}+\tfrac{3}{100}+\tfrac{3}{1000}+\cdots.$$

Here $a=\frac{3}{10}$, $r=\frac{1}{10}$, and

$$s_\infty=\frac{\frac{3}{10}}{1-\frac{1}{10}}=\frac{\frac{3}{10}}{\frac{9}{10}}=\tfrac{1}{3}.$$

19. $.222\cdots$.

20. $.123123123\cdots$.

21. $.656565\cdots$.

22. $.919191\cdots$.

23. $.1111\cdots$.

24. $.999\cdots$.

25. $415415415\cdots$.

26. $3.78223223223\cdots$.

27. $15.2555\cdots$.

ADVANCED ALGEBRA

CHAPTER XVI

COMPLEX NUMBERS

133. The imaginary unit. When we approached the solution of quadratic equations (section 65) we saw that the equation $x^2 = 2$ was not solvable if we were at liberty to use only rational numbers, but that we must introduce an entirely new kind of number (in this case $\sqrt{2}$) if we wished to solve this equation. The excuse for introducing such numbers was not that we needed them as a means for more accurate measurement — the rational numbers are entirely adequate for all mechanical purposes — but that they are a *mathematical* necessity if we propose to solve equations of the type given.

A similar situation demands the introduction of still other numbers. If we seek the solution of

$$x^2 = -1, \tag{1}$$

we observe that there is no rational number whose square is equal to -1. We may write the symbol $\sqrt{-1}$, but its meaning must be somewhat remote from that of $\sqrt{2}$, for in the latter case we have a process by which we can extract the square root and get a number whose square is as nearly equal to 2 as we may desire. This is not possible in the case of $\sqrt{-1}$. In fact this symbol differs from 1 or any real number not merely in degree but in kind. One cannot say that $\sqrt{-1}$ is greater or less than any particular real number, any more than one can compare the magnitude of a quart and of an inch.

The expression $\sqrt{-1}$ is symbolized by i and is called the **imaginary unit.** The term "imaginary" is perhaps too firmly established in mathematical literature to warrant its discontinuance. It should be kept in mind, however, that it is really no more and no less imaginary than the negative numbers or the irrational numbers. So far as we have gone it is merely something which is found to satisfy equation (1). When, however, we have defined the various operations on it and ascribed to it the various characteristic properties of numbers, we shall be justified in calling it a number.

Just as we built up from the unit 1 a system of real numbers, so we build up from $\sqrt{-1} = i$ a system of imaginary numbers. The fact that we cannot measure $\sqrt{-1}$ on a scale should cause no more confusion than our inability to measure $\sqrt[3]{2}$ with exactness on a scale. Just as we were able to operate with irrational numbers as readily as with integers when we had defined what we meant by the four operations on them, so will imaginaries become indeed numbers with which we can work when we have defined the corresponding operations on them.

134. Addition and subtraction of imaginary numbers. In defining the operations with imaginaries we follow as far as possible the analogy of the corresponding operations on integers and other real numbers. Thus we write

$$0 = 0\,i,$$

$$i + i = 2\,i,$$

$$i + i + i = 3\,i,$$

and

$$\underbrace{i + i + i + \cdots + i}_{n \text{ terms}} = ni. \qquad \text{(I)}$$

Thus in general we write $a\sqrt{-1} = ai$ where a represents any real number. Consistently with section 75 we write

$$\pm\sqrt{-a^2} = \pm\sqrt{a^2(-1)} = \pm\sqrt{a^2}\sqrt{-1} = \pm\,ai. \qquad \text{(II)}$$

We speak of a positive or negative imaginary according as the radical sign is preceded by a positive or a negative sign.

We also define addition and subtraction of imaginaries as follows:

$$ai \pm bi = (a \pm b)i, \tag{III}$$

where a and b are any real numbers.

135. Multiplication and division of imaginaries. We have virtually defined the multiplication of imaginaries by real numbers in formula (I). Consistently with section 73, we define

$$\sqrt{-1} \cdot \sqrt{-1} = ii = i^2 = -1.$$

Thus

$$\sqrt{-a} \cdot \sqrt{-b} = \sqrt{a} \cdot \sqrt{b}\, ii = (-1)\sqrt{ab} = -\sqrt{ab}. \tag{1}$$

The law of signs in the multiplication of imaginaries may be expressed verbally in the following

Principle. The product of imaginaries with like signs before the radical is a negative real number. The product of imaginaries with unlike signs is a positive real number.

For instance,

$$-\sqrt{-4} \cdot \sqrt{-9} = -2 \cdot 3 \cdot i^2 = 6.$$

We also note that $i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i, \cdots.$

And in general, $i^{4n+k} = i^k$, where $k = 0, 1, 2, 3$.

We define the division of imaginaries as follows:

$$\sqrt{-a} \div \sqrt{-b} = \frac{\sqrt{a} \cdot i}{\sqrt{b} \cdot i} = \sqrt{\frac{a}{b}}.$$

In operating with imaginary numbers, a number of the form $\sqrt{-a}$ should always be written in the form $\sqrt{a}\, i$ before the operation is performed. This avoids the temptation to make the following error, which would seem justified by section 74 unless we had defined the multiplication of imaginaries differently in (1) above.

$$\sqrt{-a} \cdot \sqrt{-b} = \sqrt{(-a)(-b)} = \sqrt{ab}.$$

EXERCISES

Simplify the following:

1. $\sqrt{-3} \cdot \sqrt{-27}$.

Solution. $\sqrt{-3} \cdot \sqrt{-27} = \sqrt{3} \cdot i \cdot \sqrt{27} \cdot i = \sqrt{81} \cdot i^2 = 9(-1) = -9$.

2. $\frac{1}{i^7}$.

Solution. $\frac{1}{i^7} = \frac{i}{i^8} = \frac{i}{(i^4)^2} = \frac{i}{1} = i$.

3. i^{25}.

4. i^{12}.

5. i^9.

6. $\sqrt{-4}$.

7. $\sqrt{-16}$.

8. $3\,i \cdot 5\,i$.

9. $\sqrt{-x^{6a}}$.

10. $\sqrt{-8\,x^2y^4}$.

11. $\sqrt{-m^8}$.

12. $\sqrt{3} \cdot \sqrt{-243}$.

13. $\sqrt{-10} \cdot \sqrt{-15}$.

14. $\sqrt{-2} \cdot \sqrt{-32}$.

15. $\frac{1}{i^5}$.

16. $\frac{1}{i^3}$.

17. $\frac{1}{i}$.

18. $\frac{\sqrt{-9}}{\sqrt{-3}}$.

19. $\frac{\sqrt{-x}}{\sqrt{-x^2}}$.

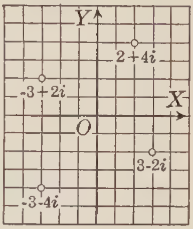
20. $\sqrt{-i^2}$.

21. $\sqrt{-i^4}$.

136. Complex numbers. The solution of the quadratic equation with negative discriminant (section 106) affords us an expression which consists of a real number connected with an imaginary number by a + or a − sign. Such an expression is called a **complex number.** It consists of two parts which are of different kinds, the real part and the imaginary part. Thus $6 + 4\,i$ means 6 1's + 4 i's. Obviously, to any pair of real numbers (x, y) corresponds the complex number $x + iy$, and conversely.

137. Graphical representation of complex numbers. We have represented all real numbers on a single straight line. When we wished to represent two numbers simultaneously (section 57), we made use of the plane, and assumed a one-to-one correspondence between the points on the plane and the pairs of numbers (x, y). The general complex number $x + iy$

depends on the values of the independent real numbers x and y, and may then be properly represented by a point on a plane. We represent real numbers on the x-axis, imaginary numbers on the y-axis, and the complex number $x + iy$ by the point (x, y) on the plane. Thus the complex numbers $-3 + 2i$, $2 + 4i$, $3 - 2i$, $-3 - 4i$ are represented by points on the plane as shown in the figure.



138. Equality of complex numbers. We define the two complex numbers $a + bi$ and $c + di$ to be equal when and only when $a = c$ and $b = d$.

Symbolically, $a + bi = c + di$

when and only when $a = c$ and $b = d$.

The definition seems reasonable, since 1 and i are different in kind, and we should not expect any real multiple of one to cancel any real multiple of the other.

Similarly, if we took, not abstract expressions as 1 and i for units, but concrete objects such as houses and streets, we should say that

a houses + b streets = c houses + d streets

when and only when $a = c$ and $b = d$.

Principle. When two numeric expressions involving imaginaries are equal to each other, we may equate real parts and imaginary parts separately.

The graphical interpretation of the definition of equality is that equal complex numbers are always represented by the same point on the plane.

From the definition given we see that $a + bi = 0$ when and only when $a = b = 0$.

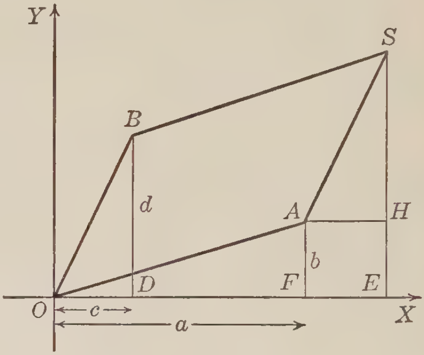
139. Addition and subtraction. By applying the assumptions just made we obtain the following symbolic expression for the operations of addition and subtraction of any two complex numbers $a + bi$ and $c + di$:

$$a + bi \pm (c + di) = a \pm c + i(b \pm d).$$

Rule. *To add (subtract) complex numbers, add (subtract) the real and imaginary parts separately.*

140. Graphical representation of addition. We now proceed to give the graphical interpretation of the operations of addition and subtraction.

Theorem. *The sum of two numbers $A = a + bi$ and $B = c + di$ is represented by the fourth vertex of the parallelogram formed on OA and OB as sides.*



Let $OASB$ be a parallelogram.

Draw $ES \perp OE$,
$AH \perp ES$,
$BD\ (= d) \perp OE$.

Then $\triangle AHS = \triangle ODB$, since their sides are parallel, and $OB = AS$.

Thus $DB = HS = d$,
$OD = AH = c$,

$$ES = EH + HS = b + d,$$

and
$$OE = OF + FE = a + c.$$

Hence S has coördinates $(a + c, b + d)$ and represents the sum of A and B, by section 139.

EXERCISES

1. Show that the difference $A - B$ of two numbers $A = a + bi$ and $B = c + di$ is represented by the extremity D of the line OD drawn from the origin parallel and equal to the diagonal BA of the parallelogram formed on OA and OB as sides.

Represent graphically the following expressions:

2. $1 + i$.

3. $3 - 2i$.

4. $i - 3$.

5. $-7 + 4i$.

6. $2 + 3i$.

7. $(i - 1) + (3 + i)$.

8. $(3 - i) - (4 - 2i)$.

9. $(1 + i) + (2 - 3i)$.

10. $(12 - 3i) - (10 - 5i)$.

11. $(2 - 3i) - 2(1 - i)$.

12. $4(7 - 5i) + (6 + 8i)$.

13. $3(i + 2) - 4(2i - 1)$.

14. $(-i - 4) - (-2i - 5)$.

15. $(16 - 15i) - (15 - 16i)$.

Solve the following for x and y by applying section 138:

16. $3x + 2iy = 15 - 12i$.

17. $x + 3iy + 2i = i + 7x - 14i$.

18. $3ix - y = 16 - 9i$.

19. $4x + 3iy = 12 + 18i$.

20. $2x - i = 7x + 2 - (2y + 4)i$.

21. $11x + 3iy - 2 = 20 - 9i$.

141. Multiplication of complex numbers. We multiply complex numbers as follows:

Rule. To multiply the complex number $a + bi$ by $c + di$, proceed as if they were real binomials, keeping in mind the laws for multiplying imaginaries.

Thus

$$\begin{array}{l} \quad a + bi \\ \underline{\quad c + di \qquad\qquad\qquad} \\ ac + bci + adi + bdi^2 = ac - bd + (bc + ad)i. \end{array}$$

142. Conjugate complex numbers. Complex numbers that differ only in the sign of their imaginary parts are called *conjugate* complex numbers, or *conjugate* imaginaries.

Theorem. The sum and the product of conjugate complex numbers are real numbers.

Thus

$$a + bi + a - bi = 2a,$$
$$(a + bi)(a - bi) = a^2 + b^2.$$

143. Division of complex numbers. The quotient of two complex numbers may now be expressed as a single complex number.

Rule. To express the quotient $\dfrac{a + bi}{c + di}$ in the form $x + yi$, rationalize the denominator, using as a rationalizing factor the conjugate of the denominator.

Thus
$$\begin{aligned}\frac{a+bi}{c+di}&=\frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}\\&=\frac{ac+bd-i(ad-bc)}{c^2+d^2}\\&=\frac{ac+bd}{c^2+d^2}-i\frac{ad-bc}{c^2+d^2}.\end{aligned}\tag{1}$$

The four operations of addition, subtraction, multiplication, and division of complex numbers have been so defined that if the imaginary parts of the complex numbers vanish and the numbers become real, the expression defining any operation on the complex numbers reduces to one defining the same operation on the real part of the numbers.

Thus in (1) above, if $b=d=0$, the expression reduces to $\frac{a}{c}=\frac{a}{c}$.

EXERCISES

Perform the indicated operations:

1. $(3+2\sqrt{-3})(5-3\sqrt{-2})$.

Solution.
$$\begin{array}{rl}3+2\sqrt{-3}= & 3+\ 2i\sqrt{3}\\ 5-3\sqrt{-2}= & \underline{5-\ 3i\sqrt{2}\qquad\qquad\qquad\qquad}\\ & 15+10i\sqrt{3}-9i\sqrt{2}+6\sqrt{6}\end{array}$$

2. $7\div(\sqrt{2}-i\sqrt{5})$.

Solution.
$$\begin{aligned}\frac{7}{\sqrt{2}-i\sqrt{5}}&=\frac{7(\sqrt{2}+i\sqrt{5})}{(\sqrt{2}-i\sqrt{5})(\sqrt{2}+i\sqrt{5})}\\&=\frac{7(\sqrt{2}+i\sqrt{5})}{2+5}\\&=\sqrt{2}+i\sqrt{5}.\end{aligned}$$

3. $(2+i)^2$.

4. $(1+2i)^4$.

Hint. Develop by the Binomial Theorem.

5. $(a+ib)^3$.

6. $(3-7i)^5$.

7. $(a-bi)^6$.

8. $(\sqrt{2}+\sqrt{-3})^2$.

9. $(\sqrt{3}+5\sqrt{-4})^5$.

10. $(ix-y)^3$.

11. $(\sqrt{x-i})(\sqrt{x+i})$.

12. $(\sqrt{3}+i\sqrt{5})(\sqrt{5}+i\sqrt{3})$.

13. $(\sqrt{x}+i\sqrt{y})(\sqrt{x}-i\sqrt{y})$.

14. $(2\sqrt{8}-5i\sqrt{3})(\sqrt{2}+2i\sqrt{3})$.

15. $\frac{2}{3\,i}$.

16. $\frac{1}{1+i}$.

17. $\frac{1+i}{1-i}$.

18. $\frac{3}{\sqrt{3}+\sqrt{-2}}$.

19. $\frac{(a+i)^2}{(a-i)^2}$.

20. $\left(\frac{1-i\sqrt{2}}{1-i\sqrt{3}}\right)^2$.

21. $\frac{2}{2+i\sqrt{2}}$.

22. $\frac{5}{7-5\,i\sqrt{3}}$.

23. $\frac{8}{\sqrt{8}-i\sqrt{2}}$.

24. $\frac{m-i\sqrt{-3}}{m+i\sqrt{-3}}$.

25. $\left(\frac{13}{5+8\,i}\right)^2$.

26. $\frac{\sqrt{-x}+\sqrt{-y}}{\sqrt{-x}-\sqrt{-y}}$.

27. $\frac{\sqrt{6}-i\sqrt{2}}{\sqrt{6}+i\sqrt{2}}$.

28. $\left(\frac{2+i\sqrt{2}}{3}\right)^3$.

29. $\frac{c+di}{a-bi}+\frac{c-di}{a+bi}$.

30. $\frac{2}{9+i\sqrt{-2}}$.

31. $\frac{25}{1+2\sqrt{-3}}$.

32. $\frac{1}{(a+i)^2}-\frac{1}{(a-i)^2}$.

33. $\frac{\sqrt{1+a}+i\sqrt{1-a}}{\sqrt{1+a}-i\sqrt{1-a}}+\frac{\sqrt{1-a}+i\sqrt{1+a}}{\sqrt{1-a}-i\sqrt{1+a}}$.

34. Find three roots of the equation $x^3-8=0$, and represent the roots as points on the plane.

35. Find four roots of the equation $x^4-16=0$, and represent the roots as points on the plane.

36. Find six roots of the equation $x^6-64=0$, and represent the roots as points on the plane. Show graphically that the sum of the six roots is zero.

37. Find three roots of the equation $x^3-27=0$, and represent the roots as points on the plane. Show graphically that the sum of the three roots is zero.

CHAPTER XVII

THEORY OF EQUATIONS

144. Equation of the *n*th degree. Any rational integral equation in one variable in which the coefficients are rational numbers can be put in the form

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \tag{1}$$

where a_0 is positive and $a_0, \cdots, a_n$ are all integers.

The symbol $f(x)$ is read "f of x" and is merely an abbreviation for the right-hand member of the equation. Often we wish to replace x in the equation by some constant, as b, -2, or 0. We may symbolize the result of this substitution by $f(b)$, $f(-2)$, or $f(0)$.

Thus $$f(b) = a_0b^n + a_1b^{n-1} + \cdots + a_n.$$

We may symbolize other expressions similarly by $F(x)$, $Q(x)$, etc.

When we speak of an equation in one variable, we assume that it is in the form of (1). This equation is also written in the form

$$x^n + b_1x^{n-1} + \cdots + b_n = 0, \tag{2}$$

where $$b_1 = \frac{a_1}{a_0},\quad b_2 = \frac{a_2}{a_0}, \cdots, \text{ and } b_n = \frac{a_n}{a_0}.$$

The b's are integers when $a_1, \cdots, a_n$ are multiples of a_0.

In elementary algebra no general method for finding the roots of an equation of higher degree than the second has been given. By the method of factoring, a few equations of higher degree have been solved. For example, the roots of the equations $x^4 - 1 = 0$ and $x^3 - 1 = 0$ may be found in this way. The present chapter is mainly devoted to the problem of finding a method for obtaining the roots of any equation of any degree, either exactly or to any desired accuracy, and to the discovery of a few of the less complex and more useful properties of equations.

145. Remainder theorem. We now prove the following important fact:

Theorem. *When $f(x)$ is divided by $x - c$, the remainder is $f(x)$ with c substituted in place of the variable x.*

We may first illustrate this theorem by a numeric example. Let $f(x) = x^3 - 3x^2 + x + 1$, and let $c = 2$. We will then show that when this expression of the third degree is divided by $x - 2$, the remainder of the division is

$$f(2) = 2^3 - 3(2)^2 + 2 + 1 = -1.$$

Now
$$\begin{array}{r|rrrr|l}
\underline{x-2} & x^3 & -3x^2 & +x & +1 & \underline{x^2-x-1}\\
 & \underline{x^3} & \underline{-2x^2} & & & \\
 & & -x^2 & +x & & \\
 & & \underline{-x^2} & \underline{+2x} & & \\
 & & & -x & +1 & \\
 & & & \underline{-x} & \underline{+2} & \\
 & & & & -1 &
\end{array}$$

Evidently the remainder is -1.

To prove the theorem in general terms we divide the equation (1) by $x - c$. Let R be the remainder, which must (section 20) be of lower degree in x than the divisor; that is, in this case, since $x - c$, the divisor, is of degree one, R must be a constant and not involve x at all. Let the quotient, which is of degree $n - 1$ in x, be represented by $Q(x)$.

Then $$f(x) \equiv Q(x)(x - c) + R. \tag{1}$$

Since this equation is an *identity*, it is *always satisfied* whatever numeric value x may have (section 52).

Let $x = c$. Then

$$f(c) = a_0c^n + a_1c^{n-1} + \cdots + a_n = Q(c)(c - c) + R.$$

But, since $c - c = 0$, $Q(c)(c - c) = 0$,

and $$f(c) = R.$$

Corollary. *If c is a root of $f(x) = 0$, then $x - c$ is a factor of the left-hand member.*

For, if c is a root of the equation, it reduces the left-hand member to zero when substituted for x. That is, $f(c)=0$. Thus by the previous theorem we have

$$f(c)=a_0c^n+a_1c^{n-1}+\cdots+a_n=R=0.$$

Hence from (1), $\quad f(x)=Q(x)(x-c).$

146. Synthetic division. In order to graph by the method of section 57 the equation

$$y=a_0x^n+a_1x^{n-1}+\cdots+a_n,$$

when the a's are replaced by integers, we should be obliged to substitute laboriously for x successive integers and to find the corresponding values of y. For large values of n this would involve considerable computation. We can make use of the preceding theorem to lighten this labor. The object is to find, with the least possible computation, the values of y which correspond to the values of x which we wish to include in the graph. By the preceding theorem the value of $f(x)$, when x is replaced by c, is the remainder obtained when the polynomial $f(x)$ is divided by $x-c$. We are therefore seeking a convenient method of computing these remainders for various values of c. As an illustration, let

$$f(x)=2\,x^4-3\,x^3+x^2-x-9, \text{ and } c=2.$$

By long division, we have

$$\begin{array}{rrrrrrl}
\underline{x-2}| & 2\,x^4 & -\,3\,x^3 & +\ \ x^2 & -\ \ x & -\ \ 9| & \underline{2\,x^3+x^2+3\,x+5}\\
 & \underline{2\,x^4} & \underline{-\,4\,x^3} & & & & \\
 & & 1\,x^3 & +\ \ x^2 & & & \\
 & & \underline{1\,x^3} & \underline{-\,2\,x^2} & & & \\
 & & & 3\,x^2 & -\ \ x & & \\
 & & & \underline{3\,x^2} & \underline{-\,6\,x} & & \\
 & & & & 5\,x & -\ \ 9 & \\
 & & & & \underline{5\,x} & \underline{-\,10} & \\
 & & & & & +\ \ 1 &
\end{array}$$

Therefore the remainder, $+1$, $=f(2)$.

We can abbreviate this process by observing the following facts. Since x is here only the carrier of the coefficient, we may omit writing it. Also we need not rewrite the first number of the partial product, as it is only a repetition of the number directly above it in heavy type. Our process now assumes the form

$$\begin{array}{lrrrrrl}
\underline{1-2}\,| & \mathbf{2} & -3 & +1 & -1 & -9\,| & \underline{2+1+3+5} \\
 & & \underline{-4} & & & & \\
 & & \mathbf{+1} & & & & \\
 & & & -2 & & & \\ \cline{3-4}
 & & & \mathbf{+3} & & & \\
 & & & & -6 & & \\ \cline{4-5}
 & & & & \mathbf{+5} & & \\
 & & & & & -10 & \\ \cline{5-6}
 & & & & & \mathbf{+\ 1} &
\end{array}$$

Since the minus sign of the 2 changes every sign in the partial product, if we replace -2 by $+2$ we may *add* the partial product to the number in the dividend instead of subtracting. This is also desirable since the number which we are substituting for x is 2, not -2. Thus, bringing all our figures on one line and placing the number substituted for x at the right hand, we have

$$\begin{array}{rrrrrl}
2 & -3 & +1 & -1 & -9\,| & \underline{2} \\
 & +4 & +2 & +6 & +10 & \\ \cline{1-5}
2 & +1 & +3 & +5 & +\ 1 &
\end{array}$$

We observe that the figures in the lower line, 2, 1, 3, 5, up to the remainder are the coefficients of the quotient

$$2x^3 + x^2 + 3x + 5.$$

Rule for synthetic division. *Write the coefficients of the polynomial in order, supplying 0 when a coefficient is lacking.*

Multiply the number to be substituted for x by the first coefficient, and add (algebraically) the product to the next coefficient.

Multiply this sum by the number to be substituted for x, add to the next coefficient, and proceed until all the coefficients are used. The last sum obtained is the remainder and also the value of the polynomial when the number is substituted for the variable.

EXERCISES

Perform by synthetic division the following divisions:

1. $7x^3 - 26x^2 + 17x - 6$ by $x - 3$.
2. $5x^3 - 3x^2 - x - 1$ by $x - 1$.
3. $15x^3 - 63x^2 + 17x - 20$ by $x - 4$.
4. $2x^3 + 3x^2 - 5x + 12$ by $x + 3$.
5. $x^3 + x^2 - 1$ by $x - 1$.
6. $3x^4 - 8x^3 - 50x^2 - 57x - 18$ by $x^2 - 5x - 6$.

HINT. $x^2 - 5x - 6 = (x - 6)(x + 1)$. Hence divide first by $(x - 6)$, and then divide the quotient by $(x + 1)$.

7. $x^6 - 1$ by $x^2 + 2x + 1$.
8. $x^4 - 2x^2 - 2x - 4$ by $x^2 - x - 2$.
9. $x^5 - 2x^4 - 5x^3 + 10x^2 + 3x - 6$ by $x - 2$.
10. $7x^4 - 17x^3 - 17x^2 + 19x - 12$ by $x - 3$.
11. $2x^4 + 13x^3 + 31x^2 + 38x + 24$ by $x^2 + 5x + 6$.
12. $2x^4 - 25x^2 + 25x - 12$ by $x^2 + x - 12$.

147. Graphing of equations. We can now form the table of values necessary to graph an equation of the type

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = y.$$

EXAMPLE

Plot $x^3 - 3x^2 + x - 4 = y$.

Solution.

x	4	3	2	1	0	-1	-2
y	16	-1	-6	-5	-4	-9	-26

$$\begin{array}{l|l} 1 - 3 + 1 - \;\;4 & \underline{4} \\ \underline{\;\; + 4 + 4 + 20} & \\ 1 + 1 + 5 + 16 & \end{array}$$

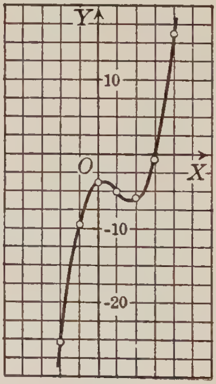
$$\begin{array}{l|l} 1 - 3 + 1 - 4 & \underline{3} \\ \underline{\;\; + 3 + 0 + 3} & \\ 1 + 0 + 1 - 1 & \end{array}$$

$$\begin{array}{l|l} 1 - 3 + 1 - 4 & \underline{2} \\ \underline{\;\; + 2 - 2 - 2} & \\ 1 - 1 - 1 - 6 & \end{array}$$

$$\begin{array}{l|l} 1 - 3 + 1 - 4 & \underline{1} \\ \underline{\;\; + 1 - 2 - 1} & \\ 1 - 2 - 1 - 5 & \end{array}$$

$$\begin{array}{l|l} 1 - 3 + 1 - 4 & \underline{-1} \\ \underline{\;\; - 1 + 4 - 5} & \\ 1 - 4 + 5 - 9 & \end{array}$$

$$\begin{array}{l|l} 1 - 3 + \;\;1 - \;\;4 & \underline{-2} \\ \underline{\;\; - 2 + 10 - 22} & \\ 1 - 5 + 11 - 26 & \end{array}$$



The x-value of the point on a curve where it crosses the x-axis is a value of x for which y is zero. Hence such values of x are the roots of the equation $y = f(x) = 0$.

By an inspection of the graph, it appears that the curve intersects the x-axis in but one place. In this case the intersection corresponds to the one real root of the equation where $y = 0$; its value is about 3.1. As we shall see later, the constants of this equation can be changed so as to cause the graph to intersect the x-axis in three points, thus causing three roots of the equation to become real.

148. Extent of the table of values. Since the object of plotting a curve is to obtain information regarding the roots of its equation, those parts of the curve beyond all crossings of the x-axis are of no interest for the present purpose. Hence it is desirable to know when a table of values has been formed extensive enough to afford a plot which includes all the real roots. If for all values of x greater than a certain number the curve lies wholly above the axis, there are no real roots greater than that value of x.

By inspection of the preceding example it appears that if for a given value of x the signs of the partial remainders are all positive, thus affording a positive value of y, any greater value of x will afford only positive partial remainders and hence only positive values of y.

Thus, *when all the partial remainders are positive, no greater positive value of* x *need be substituted.*

Similarly, *when the partial remainders alternate in sign, beginning with the coefficient of the highest power of* x, *no value of* x, *greater negatively, need be substituted.*

In plotting, if the table of values consists of values that are large or are so distributed that the plot would not be well proportioned if one space on the paper were taken for each unit, a scale should be chosen which will give a graph of good proportions, that is, so that all the portions of the curve between the extreme roots will appear on the paper, and the curvatures shall not be too abrupt to form a graceful curve. This was done, for example, in the figure of section 147.

EXERCISES

Plot and measure, to two significant figures, the values of the real roots of the following equations:

1. $x^3 + 3x - 4 = y$.
2. $7x^3 + 9x^2 - 5 = y$.
3. $x^3 + 3x + 14 = y$.
4. $2x^3 - 17x - 3 = y$.
5. $x^3 + 2x - 12 = y$.
6. $7x^3 - 10x - 3 = y$.
7. $x^3 + 13x + 14 = y$.
8. $x^3 - x + 6 = y$.
9. $x^4 + x^3 + x - 51 = y$.
10. $2x^3 + 3x^2 - x + 2 = y$.
11. $x^4 - 2x^2 + x - 2 = y$.
12. $2x^4 - 8x^3 + x^2 - 2x + 3 = y$.
13. $12x^3 - 15x^2 + 10x + 3 = y$.
14. $x^4 + 2x^3 - x^2 + 3x - 6 = y$.
15. $2x^4 + x^3 - 3x^2 + x - 12 = y$.
16. $-x^4 + x^3 - x + 3 = y$.
17. $13x^4 + 21x^3 - 6x^2 - 7x - 10 = y$.
18. $6x^4 + 7x^3 + 8x^2 - 10 = y$.

149. Roots of an equation. In the case of the linear and quadratic equations we have been able to find an explicit value of the roots in terms of the coefficients. Such processes are practically impossible in the case of most equations of higher degree. In fact, the proof that any equation possesses a root lies beyond the scope of this book, and we must therefore make the following

Assumption. *Every equation possesses at least one root.*

This is equivalent to the assumption that there is a number, rational, irrational, or complex, which satisfies any given equation.

150. Number of roots. We determine the exact number of roots of an equation by the following

Theorem. *Every equation of degree n has n roots.*

Given the equation $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, where the coefficients are real numbers.

Let r_1 (see assumption) be a root of this equation. Then (section 145) $x - r_1$ is a factor of the left-hand member, and the quotient of $f(x)$ by $x - r_1$ is a polynomial of degree $n - 1$.

Suppose that $a_0x^n + a_1x^{n-1} + \cdots + a_n$
$$= a_0(x - r_1)(x^{n-1} + b_1x^{n-2} + \cdots + b_{n-1}).$$

By our assumption the quotient,

$$x^{n-1} + b_1x^{n-2} + \cdots + b_{n-1} = 0,$$

has at least one root, say r_2, which corresponds to the factor $x - r_2$. Thus

$$f(x) = a_0(x - r_1)(x - r_2)(x^{n-2} + c_1x^{n-3} + \cdots + c_{n-2}).$$

Proceeding in this way, we find successive roots and corresponding linear factors as follows:

$$f(x) = a_0(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0,$$

where the roots are $r_1, r_2, r_3, \cdots, r_n$.

REMARK. This theorem gives no information regarding how many of the roots may be real or imaginary. This depends on the particular values of the coefficients.

Corollary. *Any polynomial in x, of degree n, may be expressed as a_0 times the product of n linear factors of the form $x - r$, where r is a real or a complex number.*

It should be noted that the roots are not necessarily distinct. Several of the roots and hence several of the factors may be identical.

If $f(x)$ is divisible by $(x - r_1)^2$, that is, if $r_1 = r_2$, we say that r_1 is a double root of the equation. Similarly, if $f(x)$ is divisible by $(x - r_1)^p$, r_1 is said to be a multiple root of the order p. When we say an equation has n roots we include each multiple root counted a number of times equal to its order.

Theorem. *An equation of degree n has no more than n distinct roots.*

Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ have the roots $r_1, r_2, r_3, \cdots, r_n$. Write the equation in the form

$$a_0(x - r_1) \cdots (x - r_n) = 0.$$

If t is a root distinct from $r_1, \cdots, r_n$, it must satisfy the equation and

$$a_0(t - r_1) \cdots (t - r_n) = 0.$$

Since this numeric expression vanishes, one of its factors must vanish (section 99). But $t \neq r_1$, hence $t - r_1 \neq 0$. Similarly, no one of the binomial factors vanishes. Thus (section 99) $a_0 = 0$, which contradicts the hypothesis that the equation is of degree n.

This theorem may also be stated as follows:

Corollary I. *If an equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ of degree n is satisfied by more than n values of x, all its coefficients vanish.*

The proof of the theorem shows that if the equation has $n + 1$ roots, $a_0 = 0$. We should then have remaining an equation of degree $n - 1$, also satisfied by $n + 1$ values of x. Thus the coefficient of its highest power in x vanishes. Similarly, each of the coefficients vanishes.

Corollary II. *If two polynomials in one variable are equal to each other for every value of the variable, the coefficients of like powers of the variable are equal.*

Let $a_0x^n + a_1x^{n-1} + \cdots + a_n = b_0x^n + b_1x^{n-1} + \cdots + b_n$, for every value of x.

Transposing, $(a_0 - b_0)x^n + \cdots + a_n - b_n = 0$.

By Corollary I,

$$\begin{aligned} a_0 - b_0 &= 0, &&\text{or} & a_0 &= b_0, \\ a_1 - b_1 &= 0, &&\text{or} & a_1 &= b_1, \\ &\vdots && & &\vdots \\ a_n - b_n &= 0, &&\text{or} & a_n &= b_n. \end{aligned}$$

151. Graphical interpretation. The graphical interpretation of the theorems of the preceding section is expressed in the statement that the graph of an equation of degree n cannot cross the x-axis more than n times. Since each crossing of the x-axis corresponds to a *real* root, there will be less than n crossings if the equation has imaginary roots.

152. Imaginary roots. We now show that imaginary roots occur in conjugate pairs. This we prove in the following

Theorem. If $a + ib$ is a root of an equation with real coefficients, $a - ib$ is also a root of the equation.

If $a + ib$ is a root of the equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, then $x - (a + ib)$ is a factor (section 145). We wish to prove that $x - (a - ib)$ is also a factor, or, what amounts to the same thing, that their product

$$\begin{aligned}[x-(a+ib)][x-(a-ib)] &= [(x-a)-ib][(x-a)+ib]\\ &= (x-a)^2+b^2\end{aligned}$$

is a factor of $f(x)$. Dividing $f(x)$ by $(x - a)^2 + b^2$, we get

$$f(x) = Q(x)[(x-a)^2+b^2] + kx + k', \tag{1}$$

where k and k' are real numbers. This remainder, $kx + k'$, can be of no higher degree in x than the first, since the divisor, $(x - a)^2 + b^2$, is only of the second degree (section 20). Now this equation (1), being an identity, is true whatever value is substituted for x, as, for instance, the root of $f(x)$, $a + ib$. Substituting this value for x, we get

$$f(a+ib) = 0 = Q(a+ib)[(a+ib-a)^2+b^2] + k(a+ib) + k',$$

or (section 53 and section 5)

$$0 = 0 + ka + k' + ikb,$$

or (section 138) $$ka + k' = 0 \tag{2}$$

and $$kb = 0. \tag{3}$$

Since $b \neq 0$, $$k = 0.$$

Also from (2), $$k' = 0.$$

Hence $$kx + k' = 0.$$

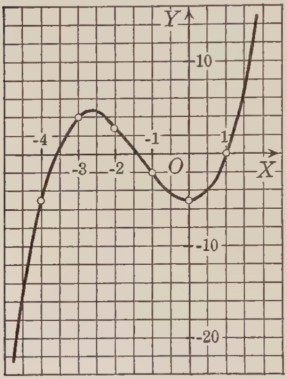
Consequently there is no remainder to the division of $f(x)$ by $(x - a)^2 + b^2$, and hence if $a + ib$ is a root of $f(x)$, $a - ib$ is also a root.

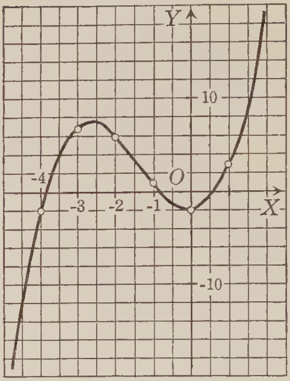
Corollary. Every equation of odd degree with real coefficients has at least one real root.

The roots cannot all be imaginary, else the degree of the equation would be even, by the preceding theorem.

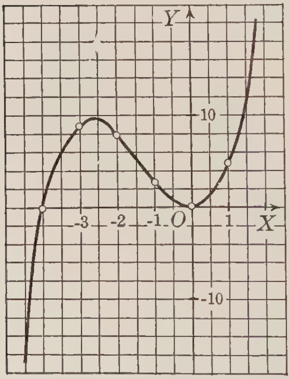
153. Graphical interpretation of imaginary roots. When we plot the equations

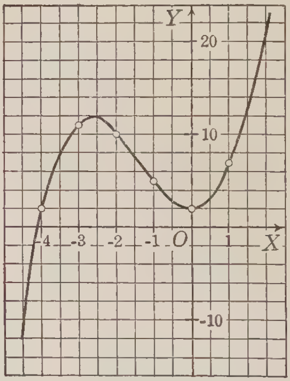
$$y = x^3 + 4x^2 - 5 \qquad (1) \qquad\qquad y = x^3 + 4x^2 - 2 \qquad (2)$$





$$y = x^3 + 4x^2 \qquad (3) \qquad\qquad y = x^3 + 4x^2 + 2 \qquad (4)$$





we see that corresponding to the increase of the constant term is a proportional elevation of the curve with respect to the x-axis. In fact, in each case the curve is the same, but the value of y is increased. In (1) and (2) we have three real roots, in (3) two of the roots coincide, and in (4) we have only one real root. As the elbow of the curve is raised and fails to intersect the x-axis *a pair* of roots cease to be real, and since a cubic equation always has three roots, *a pair* of roots become imaginary. Thus we have the

Principle. Corresponding to every elbow of the curve that does not intersect the x-axis there is a pair of imaginary roots of the equation.

The converse is not always true. It is not always possible to find as many elbows of the curve which do not meet the x-axis as there are pairs of imaginary roots.

EXERCISES

Construct the graphs of the following equations and determine from the graph how many roots are real:

1. $x^3 - 1 = y.$
2. $x^3 + 1 = y.$
3. $x^7 - 3 = y.$
4. $x^7 - x + 3 = y.$
5. $x^3 + x^2 + 1 = y.$
6. $x^5 + 3x^3 - 7 = y.$
7. $x^3 + 3x^2 - 7x + 2 = y.$
8. $x^3 + 3x^2 + 7x - 2 = y.$
9. $2x^3 + 5x^2 - 3x + 1 = y.$
10. $8x^3 + 3x - 7 = y.$

154. Relation between roots and coefficients. If we write the expression (Corollary, p. 191)

$$x^n + b_1x^{n-1} + \cdots + b_n \equiv (x - r_1)(x - r_2) \cdots (x - r_n)$$

and multiply the factors, we obtain, by equating coefficients of like powers of x, relations between the roots and the coefficients. Take, for example, $n = 3$.

$$\begin{aligned} x^3 + b_1x^2 + b_2x + b_3 &= (x - r_1)(x - r_2)(x - r_3) \\ &= x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_2r_3 + r_1r_3)x - r_1r_2r_3 = 0. \end{aligned}$$

Hence $$b_1 = -(r_1 + r_2 + r_3),$$

$$b_2 = r_1r_2 + r_2r_3 + r_1r_3,$$

and $$b_3 = -r_1r_2r_3.$$

This suggests the following

Theorem. *The coefficient of x^{n-1} is equal to the sum of the roots with their signs changed.*

The constant term is equal to the product of the roots with their signs changed.

In general the coefficient of x^{n-k} is equal to the sum of all possible products of the roots taken k at a time, their signs having been changed.

REMARK. It is to be noted that in the rule the signs of the roots are always changed before any term is formed. This does not involve any change when k is an even number, but is included in the rule for the sake of uniformity.

Corollary. *Every root of an equation is a factor of its constant term.*

155. Solution by trial. Since by the previous corollary every root of an equation is a factor of its constant term, we may in many cases test by synthetic division whether or not a given equation has integral roots. We are often able by this means to find all the roots of the equation.

Thus the integral roots of the equation

$$x^4 + 5\,x^3 + 8\,x^2 + x - 15 = 0 \tag{1}$$

must be factors of 15.

We try -3 by synthetic division:

$$\begin{array}{r|l} 1 + 5 + 8 + 1 - 15 & \underline{-3} \\ -3 - 6 - 6 + 15 & \\ \hline 1 + 2 + 2 - 5 & \end{array}$$

Thus -3 is a root of the equation (1), and the quotient of the equation by $x + 3$ is

$$x^3 + 2\,x^2 + 2\,x - 5 = 0. \tag{2}$$

If this equation has any integral root, it must be a factor of 5. We try $+5$ by synthetic division:

$$\begin{array}{r|l} 1 + 2 + 2 - 5 & \underline{5} \\ + 5 + 35 + 185 & \\ \hline 1 + 7 + 37 + 180 & \end{array}$$

Thus 5 is not a root. We try $+1$:

$$\begin{array}{r|l} 1 + 2 + 2 - 5 & \underline{1} \\ + 1 + 3 + 5 & \\ \hline 1 + 3 + 5 & \end{array}$$

Thus $+1$ is a root, and the remaining roots of (1) are the roots of

$$x^2 + 3x + 5 = 0,$$

that is,

$$\frac{-3 \pm \sqrt{-11}}{2}.$$

Hence the roots of (1) are -3, $+1$, and $\frac{-3 \pm \sqrt{-11}}{2}$.

EXERCISES

Solve by trial:

1. $x^3 - 9x - 10 = 0.$
2. $x^3 + 3x^2 - 4 = 0.$
3. $x^3 - 14x + 8 = 0.$
4. $x^3 - 17x + 26 = 0.$
5. $x^3 - 31x + 30 = 0.$
6. $x^3 - 27 = 0.$
7. $x^3 - x^2 - 10x - 8 = 0.$
8. $x^3 + 5x^2 + 10x + 8 = 0.$
9. $x^3 - 21x + 20 = 0.$
10. $x^3 + 10x^2 + 17x - 28 = 0.$
11. $x^3 - 3x^2 - 8x - 10 = 0.$
12. $x^4 + 7x^3 + 9x^2 - 16x - 28 = 0.$
13. $x^4 - 3x^3 - 19x^2 + 3x + 18 = 0.$
14. $x^4 - 13x^3 + 21x^2 + 85x + 50 = 0.$
15. $x^4 + 7x^3 - 7x^2 - 55x - 42 = 0.$
16. $x^4 - 8x^2 + 9x - 2 = 0.$
17. $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0.$
18. $x^4 - 17x^2 + 36x - 20 = 0.$
19. $x^4 - x^3 - 15x^2 + 40x - 28 = 0.$
20. $x^4 - 4x^3 - 9x^2 + 16x + 20 = 0.$

156. Formation of equations. If we know all the roots of an equation, we may form the equation in either one of two ways (see sections 104 and 154).

First method. If $r_1, r_2, \cdots, r_n$ *are the given roots, multiply together the factors* $(x - r_1) \cdots (x - r_n)$.

Second method. From the given roots form the coefficients by the theorem of section 154.

If the equation and all but one of its roots are known, that root can be found by the solution of a linear equation obtained from the coefficient of the second or the last term. If all but two of its roots are known, the unknown roots may be found by the solution of a pair of simultaneous equations formed from the same coefficients.

In the solution of the following exercises use is made of the theorem of section 154 and the various relations between the roots and the coefficients.

EXERCISES

Form equations having the following roots, by both methods of section 156:

1. $2, -1, 3$.

Solution.

First method. $(x-2)(x+1)(x-3) = x^3 - 4x^2 + x + 6 = 0$.

Second method. Let the equation be

$$x^3 + b_1x^2 + b_2x + b_3 = 0.$$

Then, by section 154, $b_1 = -(2 - 1 + 3) = -4$,

$$b_2 = -2 + 6 - 3 = 1,$$

and $$b_3 = (-2)1(-3) = 6.$$

The equation then is $x^3 - 4x^2 + x + 6 = 0$.

2. $3, 2, 4$.
3. $3, 3, 3, 3$.
4. $7, 6, 1, 0$.
5. $3, 0, 1, 0$.
6. $2, 0, 0, 0$.
7. $\pm\sqrt{3}, \pm 2i$.
8. $8, \pm\sqrt{2}, 3$.
9. $1 \pm \sqrt{2}, 2 \pm \sqrt{3}$.
10. $5, 3, -2$.
11. $8, -8, 1$.
12. $2, 1 \pm i$.
13. $-3, 2, 1 \pm \sqrt{3}$.
14. $2 \pm 3i, -2 \pm 3i$.
15. $7, \pm\sqrt{-1}$.
16. $1, -2, 2, -3$.
17. $\frac{1}{2}, 2, 1\frac{1}{2}, -1$.
18. $\pm\sqrt{2}, \pm i\sqrt{2}$.
19. $-1, 1 \pm i, \pm\sqrt{2}$.
20. $\pm 3, \frac{\pm\sqrt{3}}{7}$.
21. $\frac{1 \pm \sqrt{3}}{2}, \frac{-1 \pm \sqrt{3}}{2}$.
22. $5, -4, \frac{3 \pm 2\sqrt{-1}}{4}$.
23. $1, -2, \frac{5 \pm 2i}{2}$.

In Exercises 24 to 29 results should be obtained without factoring the equations.

24. The equation $x^4 - 5x^3 - 19x^2 + 29x + 42 = 0$ has two roots, -3 and $+2$. Find the remaining roots.

Solution. Let the unknown roots be a and b.

Then, by section 154, $-a - b + 3 - 2 = -5$,

and $$-6ab = 42.$$

Solving for a and b, we obtain

$$a = 7 \text{ or } -1,$$

and $$b = -1 \text{ or } 7.$$

25. The equation $x^3 - 19x - 30 = 0$ has the roots 5 and -3. Find the remaining roots.

26. The equation $x^3 - 7x - 6 = 0$ has the root 3. What are the remaining roots?

27. One of the roots of the equation $x^3 + 7x^2 + 7x - 15 = 0$ is the number 1. Find the remaining roots.

28. The numbers 1 and 2 are roots of $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$. What are the other roots?

29. The equation $x^3 - 12x^2 + 44x - 48 = 0$ has its roots in arithmetic progression. What are the roots?

30. The equations $x^3 - 3x - 2 = 0$ and $x^4 + 3x^3 - 2x^2 - 12x - 8 = 0$ have some of their roots in common. Plot both equations on the same axes and determine their common roots.

31. Find the middle term of the equation whose roots are 1, -2, 5, and -4 without determining any other term.

32. What is the last term of the equation whose roots are 3, ± 2, and $1 \pm \sqrt{-2}$?

33. One root of $x^4 - 4x^3 + 3x^2 + 2x - 6 = 0$ is $1 + i$. Find the remaining roots.

34. The equation $x^4 - 4x^3 + 2x^2 + 20x + 13 = 0$ has as one root, $3 - 2i$. What are the other roots?

Plot the following equations, determine all integral roots, and find the remaining roots by solving:

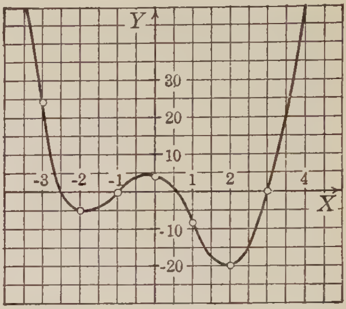
35. $x^4 - 8x^2 - 4x + 3 = 0$.

Solution.

x	4	3	2	1	0	-1	-2	-3
y	115	0	-21	-8	3	0	-5	24

Since two of the roots of this equation are seen from the graph to be $+3$ and -1, the corresponding factors of the left side of the original equation are $(x - 3)$ and $(x + 1)$. The other two roots of this equation are found by solving the equation formed by dividing the left-hand member of the original equation successively by the factors $(x-3)$ and $(x+1)$. This equation is found as follows:

$$\begin{array}{llllll}
1 & +0 & -8 & -4 & +3 & \lfloor 3 \\
 & +3 & +9 & +3 & -3 & \\
\hline
1 & +3 & +1 & -1 & 0 & \lfloor -1 \\
 & -1 & -2 & +1 & & \\
\hline
1 & +2 & -1 & 0 & &
\end{array}$$



Thus the quotient is

$$x^2+2x-1=0.$$

The roots of this equation are $-1+\sqrt{2}$ and $-1-\sqrt{2}$. These remaining roots might also be found by the method illustrated in Exercise 24.

36. $x^3-x^2-9x+9=0$.

37. $x^3-2x-4=0$.

38. $x^3+4x^2-9=0$.

39. $x^3-x^2-17x+20=0$.

40. $x^3+x^2-22x+8=0$.

41. $x^3+3x^2-2=0$.

42. $x^4+5x^3+6x^2+x-1=0$.

43. $x^4-2x^3-2x^2+3x+2=0$.

44. $x^4+x^3-10x^2-7x+15=0$.

45. How many complex roots can an equation of the seventh degree have?

46. The equation $x^3+ax^2-bx+c=0$ has two equal roots. What is the third root? What are the two equal roots?

47. The equation $x^3+x^2+x+c=0$ has one root which is the reciprocal of another. What are the roots?

48. The equation $x^3+2x^2-bx+c=0$ has two roots whose sum is zero. Find b and c.

49. The equation $x^3+3x^2-x+c=0$ has two equal roots. Find the value of c.

157. To multiply the roots by a constant. It has been stated frequently that the equation

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n=0 \qquad (1)$$

is not solvable by elementary methods when n is greater than 2. It often happens, however, that by changing its form we may obtain an equation whose roots we can find. We shall see that if an equation has *rational* roots we may find them if we change the equation as follows.

We seek to form from (1) an equation whose roots are equal to the roots of (1) multiplied by a constant factor k. Thus the equation we seek must have the roots $kr_1, kr_2, kr_3, \cdots, kr_n$. We carry out the proof, which is perfectly general, on the equation of the third order,

$$f(x) = a_0x^3 + a_1x^2 + a_2x + a_3 = 0,$$

whose roots are r_1, r_2, and r_3. The equation that we seek must have roots kr_1, kr_2, and kr_3. Since now (section 52) $f(x) = 0$ is satisfied by r, where r stands for any of the roots, that is, since $f(r) \equiv 0$, evidently $f\left(\frac{z}{k}\right) = 0$ is satisfied by kr, that is,

$$f\left(\frac{kr}{k}\right) \equiv f(r) \equiv 0.$$

Hence we obtain an equation that is satisfied by kr_1, kr_2, and kr_3 if in $f(x)$ we let $x = \frac{z}{k}$.

The required equation is then

$$f\left(\frac{z}{k}\right) = \frac{a_0z^3}{k^3} + \frac{a_1z^2}{k^2} + \frac{a_2z}{k} + a_3 = 0,$$

or, multiplying by k^3,

$$a_0z^3 + ka_1z^2 + k^2a_2z + k^3a_3 = 0.$$

This affords the following general

Rule. *To multiply the roots of an equation by a constant k, multiply the successive coefficients, beginning with the coefficient of x^{n-1}, by $k, k^2, \cdots, k^n$, respectively.*

In performing this operation, the lacking powers of x should be supplied with zero coefficients.

EXAMPLE

Multiply the roots of $4x^3 - 13x + 6 = 0$ by 3.

Solution. Multiply the coefficients in accordance with the rule above:

$$4x^3 + 0 \cdot 3x^2 - 13 \cdot 9x + 6 \cdot 27 = 0.$$

Simplifying, $$4x^3 - 117x + 162 = 0.$$

When an equation, in the form (2), section 144, has fractional coefficients, an equation may be formed whose roots are a properly chosen multiple of the roots of the original equation and whose coefficients are integers.

Corollary I. *When k is a fraction, this method serves to divide the roots of an equation by a given number.*

Corollary II. *When $k = -1$, this method serves to form an equation whose roots are equal to the roots of the original equation but opposite in sign. This is equivalent to the statement that $f(-x) = 0$ has roots equal but opposite in sign to those of $f(x) = 0$.*

EXERCISES

1. Form the equation whose roots are three times the roots of $x^4 - 5x^3 + x - 5 = 0$.

Solution. Supplying the missing term in the equation, we have

$$x^4 - 5x^3 + 0x^2 + x - 5 = 0.$$

Since $k = 3$, we have, by the rule,

$$x^4 - 3 \cdot 5x^3 + 9 \cdot 0x^2 + 27x - 81 \cdot 5 = 0,$$

or

$$x^4 - 15x^3 + 27x - 405 = 0.$$

2. Form the equation whose roots are twice the roots of $x^3 + x^2 - 3x + 2 = 0$.

3. Find an equation whose roots are one half the roots of $x^4 + 3x^3 - 5x^2 + 2x - 6 = 0$.

4. Find an equation whose roots are three quarters as great as the roots of $x^3 + 7x^2 + 3 = 0$.

By what may the roots of the following equations be multiplied so that in the resulting equation the coefficient of the highest power of x is unity and the remaining coefficients are integers? Form the equations.

5. $2x^3 - 7x^2 + 5 = 0$.

Solution. An inspection of the given equation shows us that if we can produce an equation all of whose coefficients are divisible by 2, we have satisfied the conditions of the problem.

Let $k = 2$. Then, by the rule,

$$2x^3 - 2 \cdot 7x^2 + 4 \cdot 0x + 8 \cdot 5 = 0$$

is the required equation.

Simplifying,

$$x^3 - 7x^2 + 20 = 0.$$

6. $x^3 + \frac{1}{2}x^2 - \frac{1}{4}x + 1 = 0.$

7. $x^3 - \frac{1}{27} = 0.$

8. $x^3 + \frac{a}{b}x^2 + \frac{a}{b^2}x + \frac{a}{b^3} = 0.$

9. $x^4 + \frac{1}{3}x^3 - \frac{1}{27}x + 1 = 0.$

10. $3x^3 - 5x^2 - 2x + 10 = 0.$

11. $5x^4 + 3x^2 + x - 3 = 0.$

12. $7x^3 + 2x^2 - 5x + 1 = 0.$

13. $15x^4 - 3x^3 + \frac{3}{5}x^2 + 3x - 9 = 0.$

Form equations whose roots are the negatives of the roots of the following equations:

14. $x^3 - 3x + 2 = 0.$

Solution. Supplying the missing term,

$$x^3 + 0x^2 - 3x + 2 = 0.$$

Changing signs by Corollary II,

$$x^3 - 0x^2 - 3x - 2 = 0,$$

or

$$x^3 - 3x - 2 = 0.$$

15. $x^3 - 3x^2 + 4x - 1 = 0.$

16. $x^4 + 3x^2 + 2 = 0.$

17. $x^4 + 3x^3 - x^2 + x - 1 = 0.$

18. $x^3 + 7x^2 - 5x + 2 = 0.$

19. $x^4 + 3x^3 + 9x^2 + 27x - 1 = 0.$

20. What effect does changing the sign of every term of the equation which involves an odd power of x have on the graph of the equation?

21. What is the graphical meaning of the transformation which changes the signs of the roots of an equation; that is, what relation does the graph of the equation before transformation bear to the graph of the equation after transformation, (*a*) when the degree is an even number? (*b*) when the degree is an odd number?

22. If $x^4 + 4x^3 - 3x^2 - 10x + 8 = 0$ has as two of its roots, -2 and 1, find all the roots of $x^4 - 4x^3 - 3x^2 + 10x + 8 = 0$.

23. If one root of $x^3 + 3x^2 - 16x + 12 = 0$ is 2, find the roots of $x^3 - 3x^2 - 16x - 12 = 0$.

24. If two roots of $x^4 + 2x^3 - 17x^2 - 18x + 72 = 0$ are 2 and 3, find the roots of $x^4 - 2x^3 - 17x^2 + 18x + 72 = 0$.

158. Descartes's rule of signs. A pair of successive like signs in the terms of an equation is called a **continuation of sign.** A pair of successive unlike signs is called a **change of sign.**

In the equation $2x^4 - 3x^3 + 2x^2 + 2x - 3 = 0$ there are one continuation of sign and three changes of sign. This may be seen more clearly by writing merely the signs, $+ - + + -$.

Let us now inquire what effect, if any, is noted on the number of changes of sign if an equation is multiplied by a factor of the form $x - c$, when c is positive, that is, when the number of positive roots of the equation is increased by one. Let us multiply the equation $2x^4 - 3x^3 + 2x^2 + 2x - 3 = 0$ by $x - 2$. We have then

$$\begin{array}{r}
2x^4 - 3x^3 + 2x^2 + 2x - 3 \\
x - 2 \\
\hline
2x^5 - 3x^4 + 2x^3 + 2x^2 - 3x \phantom{{}+6} \\
- 4x^4 + 6x^3 - 4x^2 - 4x + 6 \\
\hline
2x^5 - 7x^4 + 8x^3 - 2x^2 - 7x + 6
\end{array} \tag{1}$$

In this last expression the succession of signs is $+ - + - - +$, in which there are *four* changes of sign, that is, one more change of sign than in (1). If an increase in the number of positive roots *always* brings about at least an equal increase in the number of changes of sign, there must be as many changes of sign in an equation as there are positive roots, and perhaps more. This is the fact, as we now prove.

Descartes's rule of signs. *An equation $f(x) = 0$ has no more positive real roots than $f(x)$ has changes of sign.*

Illustration. In the equation of degree one, $x - 2 = 0$, there is one change of sign and one positive root. In the case of a linear equation there is no possibility of more than one change of sign, since the only changes of sign are $+ -$ and $- +$.

In the quadratic equation $x^2 + 2x + 1 = 0$ there is no change of sign, and also no positive root, since for positive values of x the expression $x^2 + 2x + 1$ is always *positive* and hence never zero. In the equation $x^2 + 2x - 3 = 0$ we have one change of sign and one positive root, $+1$. In a quadratic equation the only possible successions of sign distinct from each other are $+ + +$, $+ + -$, $+ - -$, and $+ - +$. Since the constant term is the product of the roots, the last term can be negative only when there is one positive and one negative root, provided the roots are real. Hence we have the following results. Succession $+ + +$ has no positive root; successions $+ + -$ and $+ - -$ have no more than one positive root; succession $+ - +$ may have two positive roots. Complex roots may occur with either of the successions $+ + +$ or $+ - +$.

We now prove this rule in general.

Proof of Descartes's rule. We have just seen that the rule holds for equations of degrees one and two.

Assume that the rule holds for an equation of degree m, and prove that its validity for an equation of degree $m + 1$ follows. We shall show that if we multiply an equation of degree m by $x - c$, where c is positive, thus forming an equation of degree $m + 1$, the number of changes of sign in the new equation always exceeds the number of changes of sign in the original equation by at least one. That is, the number of changes of sign increases at least as rapidly as the increase in the number of positive roots when such multiplication is made.

Let $f(x) = 0$ represent any particular equation of the nth degree. The first sign of $f(x)$ is always $+$. The remaining signs occur in successive groups of $+$ or of $-$ signs, which may contain as few as one sign in each group. If any term is lacking, its sign is taken to be the same as an adjacent sign. Thus the most general way in which the signs of $f(x)$ may occur is represented in the following table, in which the dots represent an indefinite number of signs. The multiplication of $f(x)$ by $x - c$ is represented schematically, only the signs being given.

	+ Group	− Group	+ Group	− Group	Further Groups	− Group
$f(x)$	$+\cdots+$	$-\cdots-$	$+\cdots+$	$-\cdots-$	$+\cdots+$	$-\cdots-$
$x - c$						$+-$
$x \cdot f(x)$	$++\cdots+$	$--\cdots-$	$++\cdots+$	$--\cdots-$	$++\cdots+$	$--\cdots-$
$-c \cdot f(x)$	$-\cdots-$	$-+\cdots+$	$+-\cdots-$	$-+\cdots+$	$+-\cdots-$	$-+\cdots++$
$(x-c)f(x)$	$+\pm\cdots\pm$	$-\pm\cdots\pm$	$+\pm\cdots\pm$	$-\pm\cdots\pm$	$+\pm\cdots\pm$	$-\pm\cdots\pm+$

The ± sign indicates that either the + or the − sign may occur, according to the value of the coefficients and of c. The vertical lines denote where changes of sign occur in $f(x)$. Assuming that all the ambiguous signs are taken so as to afford the least possible number of changes of sign, even then in $(x - c)f(x)$ there is a change of sign at each or between each pair of the vertical lines, and in addition, one to the right of all the vertical lines. Thus we have increased the number of positive roots by one, and we have simultaneously increased the number of changes of sign by not less than one. In fact we may increase the number of changes of sign by more than one in any case where the original equation had more changes of sign than it had positive roots.

The only possible variation that could occur in the succession of groups of signs in $f(x)$, namely, when the last group is a group of + signs, does not alter the validity of the theorem.

We may illustrate the foregoing proof by the following

EXAMPLE

Let $f(x) = x^5 + 3x^4 - 7x^3 - 6x^2 + 2x - 1$, and let $c = 3$.

$$\begin{array}{ll}
\text{Multiply} & x^5 + 3x^4 - 7x^3 - 6x^2 + 2x - 1 \quad \text{(3 changes)} \\
\text{by} & \qquad\qquad\qquad\qquad\qquad\quad x - 3 \\ \cline{2-2}
 & x^6 + 3x^5 - 7x^4 - 6x^3 + 2x^2 - x \\
 & \quad - 3x^5 - 9x^4 + 21x^3 + 18x^2 - 6x + 3 \\ \cline{2-2}
 & x^6 + 0x^5 - 16x^4 + 15x^3 + 20x^2 - 7x + 3 \quad \text{(4 changes)}
\end{array}$$

159. Negative roots. Since $f(-x)$ has roots opposite in sign to those of $f(x)$ (section 157), we can state

Descartes's rule of signs for negative roots. *The expression $f(x)$ has no more negative real roots than there are changes of sign in $f(-x)$.*

If by Descartes's rule it appears that there cannot be more than a positive roots and b negative roots, and if $a + b < n$, the degree of the equation, then there must be imaginary roots, at least $n - (a + b)$ in number.

EXERCISES

Find the maximum number of positive and negative roots, and any possible information about imaginary roots in the following equations:

1. $x^3 - 3x + 2 = 0$.

Solution. Supplying missing terms,

$$x^3 + 0x^2 - 3x + 2 = 0.$$

Writing signs only, $+ + - +$, we see that there cannot be more than two positive roots.

Writing the signs of $f(-x)$, $- + + +$, we see that there can be no more than one negative root. Since there are only three roots to this equation, there may be no complex roots whatever.

2. $x^3 - 1 = 0$.
3. $x^6 + 2 = 0$.
4. $x^4 + x^3 - 3x - 2 = 0$.
5. $x^4 + x - 1 = 0$.
6. $x^7 + 8x^5 + 2x^3 - 1 = 0$.
7. $x^3 - 5x^2 + 2x + 3 = 0$.
8. $x^5 - x^3 + x^2 - x - 1 = 0$.
9. $x^4 + 3x^3 - 5x^2 + 7x - 8 = 0$.
10. $x^7 + 3x^5 - 2x^4 + 3x^2 - x + 2 = 0$.
11. $x^6 + x^5 - 2x^4 + 3x^3 - 7x^2 + x - 1 = 0$.

160. Integral roots. In finding the rational roots of an equation, we make use of the following

Theorem. If the equation

$$x^n + a_1x^{n-1} + \cdots + a_n = 0$$

(where the a's are integers) has any rational root, such root must be an integer.

Suppose $\frac{p}{q}$ to be a fraction reduced to its lowest terms which satisfies the equation.

Hence
$$\frac{p^n}{q^n} + \frac{a_1p^{n-1}}{q^{n-1}} + \cdots + a_n = 0$$

is an identity.

Then, clearing of fractions and transposing,

$$p^n = -q(a_1p^{n-1} + \cdots + a_nq^{n-1}).$$

Thus some factor of q is a factor of p^n, that is, of p (section 65), which contradicts the hypothesis that $\frac{p}{q}$ is reduced to the lowest terms.

Thus all the rational roots of the equation are integers, which, as we know (section 154), are factors of a_n.

161. Rational roots. If we seek the rational roots of the general equation having the form

$$a_0x^n + \cdots + a_n = 0,$$

where $a_0 \neq 1$, we can multiply the roots by a properly chosen constant (section 157) and obtain an equation of the form given in the theorem of section 160, whose integral roots may easily be found by synthetic division.

EXAMPLE

What rational roots, if any, has

$$9\,x^3 - 6\,x^2 - 19\,x - 10 = 0? \tag{1}$$

Solution. Multiplying the roots by 3,

$$9\,x^3 - 18\,x^2 - 171\,x - 270 = 0.$$

Dividing this equation by 9,

$$x^3 - 2\,x^2 - 19\,x - 30 = 0. \tag{2}$$

Since by Descartes's rule equation (1) has only one positive root and may have two negative roots, we must test for the existence of all three roots. Form a table of values for (2) by synthetic division. We need try only the factors of 30 (section 154).

x	-3	-2	-1	0	1	2	3	5	6
y	-18	-8	-14	-30	-50	-68	-78	-50	0

Thus (2) has the root 6. Hence the original equation (1) has the root $6 \div 3 = 2$.

Rule. *To find all the rational roots of any equation, transform the equation so that the first coefficient is $+1$.*

Find the maximum number of positive and negative roots by Descartes's rule of signs.

Find the integral roots of this equation by trial, and the roots of the original equation by dividing the integral roots found by the constant by which the roots were multiplied.

By the theorem in section 160 we are assured that *all* the rational roots can be found in this way.

EXERCISES

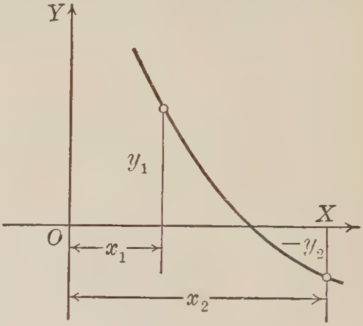
Find all the rational roots of the following equations:

1. $3x^3 = 8(x+1)$.
2. $x^3 + 5x^2 + 5x - 2 = 0$.
3. $x^3 + x^2 + 2x - 4 = 0$.
4. $6x^3 + 13x^2 + 4x - 3 = 0$.
5. $6x^3 + x^2 - 15x + 4 = 0$.
6. $3x^4 + 4x^3 - 6x^2 - 3x + 2 = 0$.
7. $4x^3 + 12x^2 + 13x + 6 = 0$.
8. $7x^3 + 17x^2 + 11x + 10 = 0$.
9. $4x^3 - 4x^2 - 25x + 25 = 0$.
10. $2x^3 - 7x^2 + 12x - 7 = 0$.
11. $6x^3 + 5x^2 - 16x - 15 = 0$.
12. $12x^3 + 20x^2 - 69x + 18 = 0$.
13. $2x^3 + 3x^2 - 11x - 6 = 0$.
14. $4x^3 + 16x^2 + 11x - 3 = 0$.
15. $27x^3 + 18x^2 - 6x - 4 = 0$.
16. $2x^4 + 3x^3 - 24x^2 - 13x + 12 = 0$.
17. $3x^4 + 11x^3 - 8x^2 - 14x + 8 = 0$.
18. $4x^4 + 8x^3 - 17x^2 - 16x + 21 = 0$.
19. $18x^4 + 33x^3 + 17x^2 + 8x + 6 = 0$.
20. $8x^4 - 4x^3 - 6x^2 + x + 1 = 0$.
21. $15x^4 + 22x^3 - 20x^2 - 7x + 2 = 0$.
22. $36x^4 - 13x^2 + 1 = 0$.
23. $3x^4 + 17x^3 + 33x^2 + 37x + 30 = 0$.

162. Irrational roots of an equation. In the preceding sections we have solved completely the problem of finding the rational roots of an equation. We now pass to the problem of finding the approximate values of the irrational roots of an equation.

For most purposes two or three significant figures are accurate enough in computing the value of the irrational roots of an equation. When no greater degree of exactness than this is required the roots may be found approximately by the graphical method given in the next section. If it is desired to make use of a numeric method which is capable of locating the root with any desired degree of accuracy, the method devised by the English schoolmaster, William George Horner (1786–1837), is most commonly employed. This method is given in section 166.

163. Location principle. If when graphing the equation $y = f(x)$ the value $x = x_1$ gives, as a corresponding value of y, a positive number equal to y_1, while the value $x = x_2$ gives the corresponding value of y negative, say equal to $-y_2$, then the point on the curve $x = x_1$, $y = y_1$ is above the x-axis, and the point on the curve $x = x_2$, $y = -y_2$ is below the x-axis. If our curve is unbroken, it must cross the x-axis at least once between the values $x = x_1$ and $x = x_2$, and hence the equation must have a root between those values of x. The shorter we can make this interval x_1 to x_2, the more accurately we can find the root of the equation. This property of unbrokenness or **continuity** of the graph of



$$y = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

we assume. We assume, then, the following

Location principle. *When for two real unequal values of x, say $x = x_1$ and $x = x_2$, the value of $y = f(x)$ has opposite signs, the equation $f(x) = 0$ has a real root between x_1 and x_2.*

Illustration. The equation $f(x) = x^3 + 3x - 5 = 0$ has a root between 1 and 2, since $f(1) = -1$ and $f(2) = 9$.

If one constructs a graph of the equation in the vicinity of the root desired, one can read from the curve an approximate value of the root. The value so obtained will probably not be exact because of the difficulty of drawing an absolutely accurate graph of the equation. If the value found is not accurate enough for the purposes in hand, one can obtain a still closer approximation by going through the same steps, using a graph on a larger scale in the immediate vicinity of the point of crossing. Just as the tenths between which the root lies were determined by synthetic division, so, taking the approximate location of the point as found above as a starting point, the hundredths between which the root lies may be found. Thus an enlarged scale may be employed to determine the value of the root to one more place of decimals.

Before beginning the search for the value of a particular root it is desirable to graph the entire curve so as to get a general idea of its behavior as it approaches the point where it crosses the x-axis.

EXAMPLE

By the graphical method find the value to two decimal places of the smallest positive root of the equation

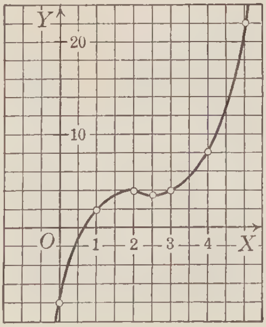
$$x^3 - 7x^2 + 16x - 8 = 0.$$

Solution. From Descartes's rule it appears that there are no negative roots for this equation.

From the graph of the equation,

$$y = x^3 - 7x^2 + 16x - 8,$$

x	0	1	2	3	4	5	$2\frac{1}{2}$
y	-8	2	4	4	8	22	3.9



it appears that there is a root near $x = .8$.

In order to determine with certainty between what tenths the root lies, the following computations suffice:

$$\begin{array}{rrrrl} 1 & -7 & +16 & -8 & \lfloor .8 \\ & +0.8 & -4.96 & +8.832 & \\ \hline 1 & -6.2 & +11.04 & +0.832 & \end{array}$$

This result shows that the root is less than .8 and that at that value of x the value of y is .832.

$$\begin{array}{rrrrl} 1 & -7 & +16 & -8 & \lfloor .7 \\ & +0.7 & -\ 4.41 & +8.113 & \\ \hline 1 & -6.3 & +11.59 & +0.113 & \end{array}$$

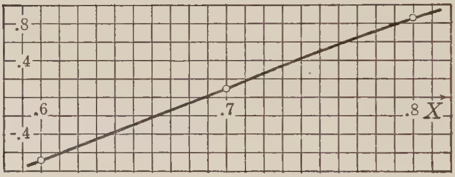
This result shows that the root is less than .7 and that for that value of x the value of y is .113.

$$\begin{array}{rrrrl} 1 & -7 & +16 & -8 & \lfloor .6 \\ & +0.6 & -\ 3.84 & +7.296 & \\ \hline 1 & -6.4 & +12.16 & -0.704 & \end{array}$$

This result shows that the root is greater than .6 and the value of y for this value of x is $-.704$.

If these results are graphed on a large scale, an inspection of the graph shows that the root is about $x = .69$.

x	.6	.7	.8
y	$-.704$	.113	.832



EXERCISES

Find the value of the real roots of the following equations to two decimal places, using the graphical method:

1. $x^3 - 2x^2 - 4x + 6 = 0.$
2. $x^2 - 2x - 12 = 0.$
3. $x^2 - 2x - 6 = 0.$
4. $x^2 - 30 = 2x.$
5. $x^2 + 6x - 104 = 0.$
6. $x^2 + 10x + 30 = 0.$
7. $x^3 + 2x^2 - 10x - 18 = 0.$
8. $x^3 - 4x^2 + 2x + 6 = 0.$
9. $x^4 - 33x^2 + 29x + 60 = 0.$
10. $2x^4 - 3x^3 + 3x^2 - 2x + 1 = 0.$

164. Diminishing the roots of an equation. In order to find the roots of an equation by Horner's method, it is necessary to be able to form an equation whose roots are equal respectively to the roots of the original equation, each diminished by a constant.

Let $$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \tag{1}$$
whose roots are $r_1, r_2, \cdots, r_n$. Let c be any constant. We seek an equation whose roots are $r_1 - c, r_2 - c, \cdots, r_n - c$.

If we let r stand for any of the roots of (1), since $f(r) = 0$ (section 145) we see that $f(z + c) = 0$ is satisfied by $r - c$, that is,
$$f(r - c + c) = f(r) = 0.$$

Thus to form the desired equation, replace x by $z + c$. We then obtain
$$f(x) = f(z + c) = a_0(z + c)^n + a_1(z + c)^{n-1} + \cdots + a_n = 0.$$

Developing each term by the binomial theorem and collecting like powers of z, we get an equation of the form
$$f(x) = F(z) = A_0z^n + A_1z^{n-1} + \cdots + A_n = 0, \tag{2}$$
where the A's involve the a's and c. This is the equation desired.

We now seek a convenient method of finding the values of the coefficients $A_0, A_1, \cdots, A_n$ when $a_0, a_1, \cdots, a_n$ and c are given numerically. Now A_n is the remainder from the division of $F(z)$ by z. But since $F(z) = f(x)$, and $z = x - c$, the remainder from dividing $F(z)$ by z is identical with the remainder from dividing $f(x)$ by $x - c$. Thus A_n is the remainder from dividing $f(x)$ by $x - c$. Furthermore, since A_{n-1} is the remainder from dividing $\dfrac{F(z) - A_n}{z}$ by z, it is also the remainder from dividing $\dfrac{f(x) - A_n}{x - c}$ by $x - c$. The process is continued for finding the other A's. We may then diminish the roots of an equation by c, observing the following

Rule. *The constant term of the new equation is the remainder from dividing $f(x)$ by $x - c$.*

The coefficient A_{n-1} in the new equation is the remainder from dividing the quotient just obtained by $x - c$.

The coefficients $A_{n-2}, A_{n-3}, \cdots, A_2, A_1, A_0$ are the remainders from dividing the successive quotients by $x - c$.

EXAMPLE

Form an equation whose roots are 3 less than the roots of $x^4 - 3x^3 + 5x^2 + 7x - 12 = 0$.

Solution. Performing the required division synthetically, we find that our work arranges itself as follows:

$$\begin{array}{rrrrrrrrrl}
1 & - & 3 & + & 5 & + & 7 & - & 12 & \underline{|3} \\
 & + & 3 & + & 0 & + & 15 & + & 66 & \\
\cline{1-9}
1 & + & 0 & + & 5 & + & \multicolumn{1}{r|}{22} & + & 54 & \\
 & + & 3 & + & 9 & + & \multicolumn{1}{r|}{42} & & & \\
\cline{1-7}
1 & + & 3 & + & \multicolumn{1}{r|}{14} & + & 64 & & & \\
 & + & 3 & + & \multicolumn{1}{r|}{18} & & & & & \\
\cline{1-5}
1 & + & \multicolumn{1}{r|}{6} & + & 32 & & & & & \\
 & + & \multicolumn{1}{r|}{3} & & & & & & & \\
\cline{1-3}
1 & + & 9 & & & & & & &
\end{array}$$

The desired equation is

$$x^4 + 9x^3 + 32x^2 + 64x + 54 = 0.$$

165. Graphical interpretation of decreasing roots. If an equation has roots c units less than those of another equation, and if c is positive, its intersections with the x-axis or with any line parallel to the x-axis are c units to the left of the corresponding intersections of the first equation. It is, in fact, the same curve, except that the y-axis is moved c units to the right. If c is negative in sign, the y-axis is moved c units to the left.

EXERCISES

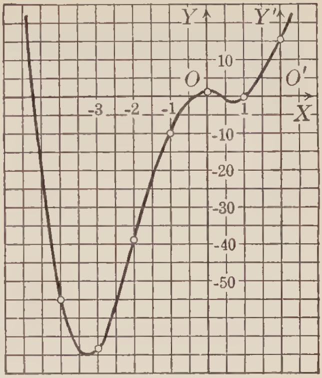
Plot, decrease the roots by c units, state the new equation, and locate the new axes.

1. $x^4 + 3x^3 - 7x^2 + 2x + 1 = 0$, $c = 2$.

Solution.

x	-5	-4	-3	-2	-1	0	.9	1	2
y	66	-55	-68	-39	-10	1	$-.03$	0	17

$$\begin{array}{rrrrrl}
1 & +\ 3 & -\ \ 7 & +\ \ 2 & +\ \ 1 & \underline{|2} \\
 & +\ 2 & +\ 10 & +\ \ 6 & +\ 16 & \\
\hline
1 & +\ 5 & +\ \ 3 & +\ \ 8 & +\ 17 & \\
 & +\ 2 & +\ 14 & +\ 34 & & \\
\cline{1-4}
1 & +\ 7 & +\ 17 & +\ 42 & & \\
 & +\ 2 & +\ 18 & & & \\
\cline{1-3}
1 & +\ 9 & +\ 35 & & & \\
 & +\ 2 & & & & \\
\cline{1-2}
1 & +\ 11 & & & &
\end{array}$$



Thus the required equation is

$$x^4 + 11\,x^3 + 35\,x^2 + 42\,x + 17 = 0.$$

The axes for this equation have been located on the adjacent graph.

2. $x^4 - 81 = 0$, $c = 3$.

3. $x^3 - 27 = 0$, $c = 1.5$.

4. $x^4 + 3\,x^2 - 1 = 0$, $c = 2$.

5. $x^3 + x^2 - 1 = 0$, $c = 1$.

6. $x^3 - 9\,x + 3 = 0$, $c = 3$.

7. $6\,x^3 + 11\,x^2 - 9\,x + 1 = 0$, $c = \frac{1}{2}$.

8. $3\,x^3 + 29\,x^2 + 60\,x + 28 = 0$, $c = -2$.

9. $x^3 + 3\,x^2 - 4\,x - 12 = 0$, $c = 2$.

10. $x^3 - 6\,x^2 - 7\,x + 60 = 0$, $c = 4.5$.

11. $x^4 + 5\,x^3 - 7\,x^2 - 41\,x - 30 = 0$, $c = 10$.

12. $x^3 + 2\,x^2 - 5\,x - 6 = 0$, $c = -3$.

13. $2\,x^3 + x^2 - 24\,x + 9 = 0$, $c = 5$.

14. $3\,x^4 + 7\,x^3 + 3\,x^2 - 3\,x - 2 = 0$, $c = 1$.

15. $x^3 - 6\,x^2 - 36\,x + 5 = 0$, $c = -2$.

16. $x^4 + 2\,x^3 - 12\,x^2 + 3\,x - 4 = 0$, $c = 1$.

17. $x^3 - 3\,x^2 + 3\,x + 1 = 0$, $c = 1$.

18. $x^4 - 12\,x^3 + 3\,x^2 + 7\,x - 5 = 0$, $c = 3$.

19. From an inspection of the results of Exercises 15–18 determine by how much it would be necessary to increase the roots of the equation $x^n + a_1x^{n-1} + \cdots + a_n = 0$ so that the coefficient of x^{n-1} will be lacking in the transformed equation.

20. Without actually solving the equation, show whether it would be possible to increase or diminish the roots of the equation $x^n + a_n = 0$ by a number which would cause the constant term of the transformed equation to vanish.

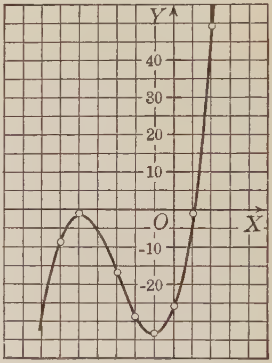
166. Approximate calculation of roots by Horner's method. We are now in a position to compute to any desired degree of accuracy the real roots of an equation. Consider, for example, the equation

$$x^3 + 9x^2 + 15x - 26 = 0. \tag{1}$$

Form the table for plotting the equation

$$x^3 + 9x^2 + 15x - 26 = y.$$

x	-6	-5	-4	-3	-2	-1	0	1	2
y	-8	-1	-6	-17	-28	-33	-26	-1	48



By the location theorem we find that a root lies between $+1$ and $+2$. To find more precisely the position of the root, we might estimate from the graph the position of the root and substitute, say, 1.1 or 1.2 and so on, until we found two values between which the root lies. We can obtain the same result with much less computation if we first diminish the roots of the equation so that the origin is at the lesser of the two integral values between which the root lies. Here we decrease the roots of (1) by 1,

$$\begin{array}{rrrrrrr|l}
1 & + & 9 & + & 15 & - & 26 & \underline{1} \\
 & + & 1 & + & 10 & + & 25 & \\
\cline{1-7}
1 & + & 10 & + & \multicolumn{1}{r|}{25} & - & 1 & \\
 & + & 1 & + & \multicolumn{1}{r|}{11} & & & \\
\cline{1-5}
1 & + & \multicolumn{1}{r|}{11} & + & 36 & & & \\
 & + & \multicolumn{1}{r|}{1} & & & & & \\
\cline{1-3}
1 & + & 12 & & & & &
\end{array}$$

The equation whose roots are decreased by 1 is

$$x^3 + 12x^2 + 36x - 1 = 0. \tag{2}$$

We know that (2) has a root between 0 and 1, since equation (1) has a root between 1 and 2. From the graph we can estimate the position of the root. Having made an estimate, say .1, it is necessary to verify the estimate and determine by synthetic division precisely between which tenths the root lies. Thus, trying .1, we obtain

$$\begin{array}{rrrrl} 1 & +\ 12 & +\ 36 & -\ 1 & \lfloor .1 \\ & +\ \ 0.1 & +\ \ 1.21 & +\ 3.721 & \\ \hline 1 & +\ 12.1 & +\ 37.21 & +\ 2.721 & \end{array}$$

which shows that the curve lies above the x-axis at the point $x = .1$. Since the value of y for $x = 0$ is -1, we know that the curve lies below the x-axis at the point $x = 0$. We consequently know that the root of (2) lies between 0 and .1. In other words, equation (1) has a root lying between 1.0 and 1.1. Since the remainder from dividing (2) by 0 is -1, and the remainder from dividing by .1 is $+2.721$, we are safe in assuming that the root of (1) lies closer to 1.0 than to 1.1.

To find the root correct to two decimal places, we should ordinarily move the origin again, this time moving it to the lesser of the two numbers between which the root of (2) lies. Since the lesser of these two numbers is zero, in this case we shall not need to move the origin at all.

Since the equation (2) has a root between 0 and .1, the value of x^3, and also of x^2, will be extremely small, and may without serious error be omitted in computing an approximate value of the next decimal place. This step leaves us with the following linear equation whose root is the approximate root of (2):

$$36\,x - 1 = 0.$$

Thus

$$x = .03.$$

This suggestion must be verified by synthetic division to determine between what hundredths a root of (2) actually lies.

$$\begin{array}{rrrrl} 1 & +\ 12 & +\ 36 & -\ 1 & \lfloor .03 \\ & +\ \ 0.03 & +\ \ 0.3609 & +\ 1.090827 & \\ \hline 1 & +\ 12.03 & +\ 36.3609 & +\ 0.090827 & \end{array}$$

Since the remainder is + .090827, we know that the curve is above the x-axis at this point, and therefore the root is less than .03. Trying .02 as a possible value, we obtain

$$\begin{array}{rrrrl} 1 & +\ 12 & +\ 36 & -\ 1 & \underline{|.02} \\ & +\ 0.02 & +\ 0.2404 & +\ 0.724808 & \\ \hline 1 & +\ 12.02 & +\ 36.2404 & -\ .275192 & \end{array}$$

which indicates that the curve is below the x-axis at the point .02. A comparison of the remainders from these two divisions shows that the root is closer to .03 than to .02. We now know that the root of equation (1) lies between 1.02 and 1.03 and is nearer to 1.03. If a greater degree of precision is desired, this process may be continued indefinitely, and the root found correct to any required number of decimal places.

The foregoing process affords the following

Rule. *Plot the equation. Apply the location principle to determine between what consecutive positive integral values a root lies.*

Decrease the roots of the equation by the lesser of the two integral values between which the root lies.

Estimate from the graph the nearest tenth to which the root of the new equation lies, and determine by synthetic division precisely the successive tenths between which the root lies.

Decrease the roots of this equation by the lesser of the two tenths between which the root lies, and estimate the root to the nearest hundredth by solving the last two terms as a linear equation.

Determine by synthetic division precisely the hundredths interval in which the root lies.

Proceed similarly to find the root correct to as many decimal places as may be desired.

The sum of the integral, tenths, hundredths, and succeeding values next less than the root in the various processes is the approximate value of the root.

To find the negative roots of an equation $f(x) = 0$, find the positive roots of $f(-x) = 0$ and change their signs.

When all the roots are real, a check on the accuracy of the computation may be found by adding the roots together. The result should approximately equal the coefficient of the second term of the original equation.

EXERCISES

Find the values of the real roots of the following equations, correct to three decimal places:

1. $x^3 + 8x^2 + 21x + 25 = 0.$ (1)

Solution. Descartes's rule shows that there are no positive roots. We will form $f(-x) = 0$ and seek its positive roots.

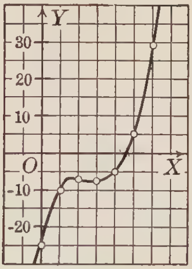
Thus $$x^3 - 8x^2 + 21x - 25 = 0. \qquad (2)$$

Plot the equation (2) set equal to y. In the figure it may be seen that there is a root at approximately 4.6.

x	0	1	2	3	4	5	6
y	− 25	− 11	− 7	− 7	− 5	5	29

Decrease the roots of (2) by 4:

$$\begin{array}{rrrrl} 1 & -8 & +21 & -25 & \underline{\lfloor 4} \\ & +4 & -16 & +20 & \\ \hline 1 & -4 & +5 & -5 & \\ & +4 & +0 & & \\ \hline 1 & +0 & +5 & & \\ & +4 & & & \\ \hline 1 & +4 & & & \end{array}$$



The equation is $x^3 + 4x^2 + 5x - 5 = 0.$ (3)
Verify .6 and .7 as roots of (3):

$$\begin{array}{rrrrl} 1 & +4 & +5 & -5 & \underline{\lfloor .6} \\ & +0.6 & +2.76 & +4.656 & \\ \hline 1 & +4.6 & +7.76 & -.344 & \end{array}$$

$$\begin{array}{rrrrl} 1 & +4 & +5 & -5 & \underline{\lfloor .7} \\ & +0.7 & +3.29 & +5.803 & \\ \hline 1 & +4.7 & +8.29 & +.803 & \end{array}$$

Thus the root is determined between .6 and .7.

Decrease the roots of (3) by .6:

$$\begin{array}{rrrrrrrl}
1 & + & 4 & + & 5 & - & 5 & \underline{|.6} \\
 & + & 0.6 & + & 2.76 & + & 4.656 & \\ \cline{1-7}
1 & + & 4.6 & + & 7.76 & - & .344 & \\
 & + & .6 & + & 3.12 & & & \\ \cline{1-5}
1 & + & 5.2 & + & 10.88 & & & \\
 & + & .6 & & & & & \\ \cline{1-3}
1 & + & 5.8 & & & & &
\end{array}$$

The equation is $x^3 + 5.8\,x^2 + 10.88\,x - .344 = 0.$ (4)

Estimate the root of (4) at $x = \dfrac{.344}{10.88} = .03$, and verify:

$$\begin{array}{rrrrrrrl}
1 & + & 5.80 & + & 10.8800 & - & .344000 & \underline{|.03} \\
 & + & .03 & + & .1749 & + & .331647 & \\ \cline{1-7}
1 & + & 5.83 & + & 11.0549 & - & .012353 &
\end{array}$$

$$\begin{array}{rrrrrrrl}
1 & + & 5.80 & + & 10.8800 & - & .344000 & \underline{|.04} \\
 & + & .04 & + & .2336 & + & .444544 & \\ \cline{1-7}
1 & + & 5.84 & + & 11.1136 & + & .100544 &
\end{array}$$

Reduce the roots of (4) by .03:

$$\begin{array}{rrrrrrrl}
1 & + & 5.80 & + & 10.8800 & - & .344000 & \underline{|.03} \\
 & + & .03 & + & .1749 & + & .331647 & \\ \cline{1-7}
1 & + & 5.83 & + & 11.0549 & - & .012353 & \\
 & + & .03 & + & .1758 & & & \\ \cline{1-5}
1 & + & 5.86 & + & 11.2307 & & & \\
 & + & .03 & & & & & \\ \cline{1-3}
1 & + & 5.89 & & & & &
\end{array}$$

The equation is $x^3 + 5.89\,x^2 + 11.2307\,x - .012353 = 0.$ (5)

Estimate the root of (5) at $x = \dfrac{.012353}{11.2307} = .001$, and verify:

$$\begin{array}{rrrrrrrl}
1 & + & 5.890 & + & 11.230700 & - & .012353000 & \underline{|.001} \\
 & + & .001 & + & .005891 & + & .011236591 & \\ \cline{1-7}
1 & + & 5.891 & + & 11.236591 & - & .001116409 &
\end{array}$$

$$\begin{array}{rrrrrrrl}
1 & + & 5.890 & + & 11.230700 & - & .012353000 & \underline{|.002} \\
 & + & .002 & + & .011784 & + & .022484968 & \\ \cline{1-7}
1 & + & 5.892 & + & 11.242484 & + & .010131968 &
\end{array}$$

Thus the root of equation (1), correct to three decimal places, is -4.631.

2. $x^4 - 27 = 0.$

3. $x^3 - x = 17.$

4. $x^3 - 18 = 0.$

5. $7\,x^4 + 3\,x^3 + 3 = 0.$

6. $x^3 + x^2 = 150.$

7. $x^3 - x - 10 = 0.$

8. $x^4 + x - 19 = 0$.
9. $x^3 - 7x + 13 = 0$.
10. $x^4 + 5x^3 + 7x^2 + x - 2 = 0$.
11. $x^4 + x^3 + x = 135$.
12. $x^3 - x + 3 = 0$.
13. $x^3 + 3x - 5 = 0$.
14. $x^3 + 2x^2 - 3x - 2 = 0$.
15. $x^3 + x^2 + x - 99 = 0$.
16. $x^3 - 5x^2 - 3x + 1 = 0$.
17. $x^4 - x^3 + x^2 - x + 66 = 0$.
18. $x^4 + 12x^3 + 6x^2 - 3x + 2 = 0$.
19. $x^3 + 5x^2 + 6x - 11 = 0$.
20. $x^3 + 4x^2 + x - 2 = 0$.
21. $3x^4 + 2x^3 + x^2 - x + 3 = 0$
22. $2x^4 - 17x^3 + 16x^2 - 10x + 5 = 0$.
23. $9x^3 + 3x^2 + x - 5 = 0$.
24. $x^4 + 3x^3 - 7x^2 + 2x - 1 = 0$.
25. $10x^3 + 30x^2 - 10x + 1 = 0$.
26. $x^3 - 7x + 7 = 0$.
27. $2x^5 + 3x^4 + 3x^3 - 5x^2 - 5x + 2 = 0$.
28. $24x^3 - 2x^2 - 17x - 5 = 0$.
29. $36x^3 + 9x^2 - 22x + 5 = 0$.
30. $x^4 - 3x^3 + x + 1 = 0$.

CHAPTER XVIII

PERMUTATIONS, COMBINATIONS, AND PROBABILITY

167. Introduction. Before dealing directly with the subject of the chapter, we must answer the question In how many distinct ways may two successive acts be performed, if the first may be performed in p ways and the second may be performed in q ways? Suppose, for example, that I can leave a certain house by any one of four doors, and can enter another house by any one of five doors, in how many ways can I pass from one house to the other? If I leave the first house by a certain door, I have the choice of all five doors by which to enter the second house. Since, however, I might have left the first house by any one of its four doors, there will be $4 \cdot 5 = 20$ ways in which I may pass from one house to the other. This fact suggests the following

Theorem. *If a certain act may be performed in p ways, and after this act is performed a second act may be performed in q ways, then the total number of ways in which the two acts may be performed is $p \cdot q$.*

With *each* of the p possible ways of performing the first act correspond q ways of performing the second act. Thus with *all* the p possible ways of performing the first act must correspond q times as many ways of performing the second act. That is, the two acts may be performed in $p \cdot q$ ways.

Unless the contrary is stated, it will be assumed that the acts referred to are independent of each other. This means that in whatever way one act is performed there is perfect freedom to select any of the possible ways of performing the second act. In the example mentioned above it is assumed that the door by which the first house is left does not affect the choice of doors by which one would enter the second house.

In the following exercises, as in all the exercises of this chapter, one should not appeal to the formula as a substitute for thinking through the conditions of the problem. In every case one should approach the idea of the problem directly, often making no use of the formula except to verify or to put in more elegant form the result that may have been reached.

PROBLEMS

1. A man eating in a restaurant has a choice of 3 meats and 4 desserts. How many meals can he order with this choice?

Solution. With each of the meat courses he may select any one of 4 desserts. This allows him 4 times as many meals as the number of meats he may select. He will accordingly be able to form 12 meals.

2. If a man has a choice of 4 railroads between Chicago and New York, and the choice of 6 steamers between New York and Liverpool, in how many ways could he travel from Chicago to Liverpool?

3. In how many ways can 2 men stop at 3 hotels so that both men do not stay at the same hotel?

4. In how many ways can 2 men stop at 4 hotels so that both men do not stay at the same hotel?

5. In how many ways can 4 men stay at 4 hotels so that no 2 men stay at the same hotel?

6. A man has 4 overcoats and 3 hats. How many combinations can he form?

7. In traveling from Boston to New York I have a choice of 6 means of transportation. In how many ways can I make the round trip so that I return by a different means from that used when I went?

8. An actor has 10 suits, 6 hats, and 4 ways of making up his face. How many different appearances has he at his disposal?

9. How many meals could a man choose if he had a choice of 2 soups, 5 meat courses, 3 salads, and 4 desserts?

168. Permutations. Each different arrangement either of all or of a part of a number of things is called a **permutation.**

Thus the digits 1 and 2 have two possible permutations, taken both at a time, namely, 12 and 21.

The digits 1, 2, and 3 have six different permutations when two are taken at a time; namely, 12, 13, 21, 23, 31, and 32. For if we take 1 for the first place, we have a choice of 2 or 3 for the second place, and we get 12 and 13. If 2 is chosen for the first place, we get 21 and 23. Similarly, we get 31 and 32. In this process it will be noticed that we can fill the first of the two places in any one of three ways; the second place can be filled in each case in only two ways. Thus, by the theorem, section 167, we expect $3 \cdot 2 = 6$ permutations of three things taken two at a time. We observe that this product $3 \cdot 2$ has, as its first factor, 3, which is the total number of things considered. The number of factors is equal to the number of things taken at a time, that is, two. This leads to the general

Theorem. *The number of permutations of n objects taken r at a time is*

$$n(n-1)\cdots(n-r+1). \tag{I}$$

This is symbolized by ${}_nP_r$.

This expression is easily remembered if one observes that the first factor is n, the total number of objects considered, and that the number of factors is r, the number of objects taken at a time. Thus ${}_7P_3 = 7 \cdot 6 \cdot 5$.

To prove this theorem, first let $r = 1$. There are evidently only n different arrangements of n objects, taking one at a time, since there is only one way of arranging a single object.

Next take two objects at a time; that is, let $r = 2$. Since there are n objects, we have n ways of filling the first of the two places. When that is filled there are $n - 1$ objects left, and any one may be used in filling the second place. Thus, by the theorem, section 167, there are, for $r = 2$, $n(n-1)$ different permutations. This verifies Formula (I) when $r = 2$.

We now assume the form (I) for $r = m$ where $m < 3$, and prove it for $r = m + 1$.

$$_nP_m = n(n-1)\cdots(n-m+1). \tag{1}$$

We can fill the first m places in $_nP_m$ different ways, since there are $_nP_m$ permutations of n things taken m at a time. This constitutes the first act (section 167). The second act consists in filling the $(m+1)$th place, which may be done in $n-m$ ways by using any one of the remaining $n-m$ objects. This can be done in $_nP_m(n-m)$ ways. Thus the number of permutations of n things taken $m+1$ at a time is

$$\begin{aligned} _nP_{m+1} &= {}_nP_m(n-m) \\ &= n(n-1)(n-2)\cdots(n-m+1)(n-m), \end{aligned}$$

which is the form that (1) assumes on replacing m by $m+1$.

Corollary. The number of permutations of n things taken all at a time is

$$\boldsymbol{_nP_n = n(n-1)\cdots 2\cdot 1 = n!}^* \tag{II}$$

Taking $r = n$ in (I), we get (II).

PROBLEMS

1. In how many ways may one arrange three books taken from a shelf containing twenty books?

Solution. There are 20 choices for the first position, 19 choices for the second position, and 18 for the third position. By section 167 this affords a total number of arrangements equal to $20 \cdot 19 \cdot 18 = 6840$ permutations.

2. In how many different ways can six books be arranged on a shelf?

3. How many five-letter code words can be formed from the alphabet of twenty-six letters?

4. How many three-letter code words may be formed if only twenty-four letters of the alphabet can be used?

5. In how many ways can the letters of the word *publisher* be arranged?

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* The symbol $n!$ stands for $1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1)n$ and is read *factorial n*.

6. How many two-digit numbers may be formed from the numerals 1 to 5, inclusive, if repetitions are not allowed?

7. How many signals may be set on a straight line, using seven signal-flags, if there are 15 flags to choose from?

8. In how many different ways may four different makes of tires be placed on an automobile if there are twenty makes of tires available?

9. I have ten books on science and five books of literature. In how many different ways can these books be placed on two shelves so that all the scientific books are on one shelf, and all the literature is on the other shelf?

HINT. See corollary, section 168, and theorem, section 167.

10. I have six sea pictures and eight landscapes. These must be hung in two rooms so that all the sea pictures are in a row in one room and all the landscapes are in a row in the other room. In how many different ways can this be done?

169. Combinations. Any group of things that is independent of the order of the constituents of the group is called a **combination.**

The committee of men Jones, Smith, and Jackson is the same as the committee Jackson, Jones, and Smith. In general, a question involving the number of groups of objects that may be formed, where the character of any group is unaltered by any change of order among its constituent parts, is a question in combinations.

Suppose, for example, that we ask how many committees of three men can be selected from six men. If the men are called A, B, C, D, E, and F, there are, by section 168, $6 \cdot 5 \cdot 4 = 120$ different arrangements or permutations of the six men in groups of three each. But while the permutations A, B, C; A, C, B; B, A, C, etc., are all distinct, the committees consisting of A, B, and C are identical. According to the corollary on page 225, there are $3! = 6$ permutations of three things taken all at a time. Hence all 6 of these permutations of the men A, B, and C afford the same committee. This is true for every distinct set of three men that we could select; that is, for the six different permutations of any three men there is only one committee. Hence the number of committees is one sixth of the total number of permutations, or

$$\frac{{}_6P_3}{3!}.$$

This leads to the general

Theorem. The number of combinations (${}_nC_r$) *of n things taken r at a time is*

$$ {}_nC_r=\frac{n(n-1)\cdots(n-r+1)}{r!}. \tag{I}$$

Since the numerator of (I) is ${}_nP_r$, we have

$${}_nC_r=\frac{{}_nP_r}{r!}.$$

Thus there are $r!$ times as many permutations as combinations.

This formula is easily remembered if one observes that there is the same number of factors in the numerator as in the denominator. Thus

$${}_{10}C_3=\frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3}.$$

Corollary. $\quad {}_nC_r={}_nC_{n-r}.$

Multiplying the numerator and denominator of (I) by $(n-r)!$,

$$\begin{aligned} {}_nC_r &= \frac{n(n-1)\cdots(n-r+1)(n-r)\cdots 2\cdot 1}{r!(n-r)!} \\ &= \frac{n!}{r!(n-r)!} \\ &= \frac{n(n-1)\cdots(r+1)}{(n-r)!} \\ &= \frac{n(n-1)\cdots[n-(n-r)+1]}{(n-r)!} \\ &= {}_nC_{n-r}. \end{aligned}$$

This corollary saves computation in some cases. For example, if we wish to compute ${}_{19}C_{17}$, it is more convenient to write ${}_{19}C_2=\frac{19\cdot 18}{1\cdot 2}=171$ than the expression for ${}_{19}C_{17}$.

PROBLEMS

1. I have ten horses from which to select a team of four. Three of the ten horses are suitable to be chosen for the two leaders but cannot be used in the other positions. How many teams can I choose from these horses?

Solution. The solution of the problem involves two distinct steps: First, the selection of two horses out of a group of three. In this case there are three combinations. Second, we must pick two horses from the remainder of the group which contains seven horses. There are twenty-one ways of doing this. By section 167, sixty-three possible teams may be formed.

2. I have twenty men and three coxswains on my rowing squad. How many eight-oared crews can I form from this squad?

3. I have a choice of five used tires and four new tires with which to fit out my automobile. The used tires cannot be used on the rear wheels, and the new tires may not be used on the front wheels. In how many ways may I equip the four wheels of my car?

4. How many straight lines can be drawn through fifteen points no three of which are in a straight line?

5. How many triangles can be formed with twenty-three points no three of which are in the same straight line?

6. How many sounds can be made from the thirteen notes of an octave on a piano if three notes must be struck simultaneously for each sound?

7. Ten people meet and wish to shake hands. How many handshakes are exchanged when everyone has shaken hands with everyone else?

8. In how many ways may p points no three of which are in the same straight line be connected by straight lines?

9. How many triangles can be formed from p points no three of which are in the same straight line?

10. In how many ways may the product $r \cdot s \cdot t \cdot w \cdot x \cdot y \cdot z$ be broken up into three factors two of which contain two letters while the third contains three letters?

11. A crew consists of eight men of whom two can row only on the right side, while one man can row only on the left side. How many arrangements can be made with the men in this crew?

12. How many baseball nines can be formed from eighteen men of whom three can pitch and play no other position, five can only catch, and the rest can play any other position?

13. Two college buildings face each other across the campus. One building has four doors, while the other has five doors. Each door in one building is connected by a path to each door in the other building. How many paths are there?

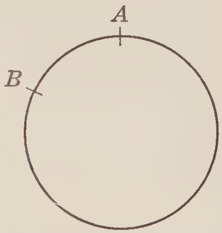
14. From twenty consonants and six vowels, how many four-letter words can be formed, using one and only one vowel in each word?

15. From a pack of fifty-two cards, how many different hands of five cards may be dealt?

170. Circular permutations. By circular permutations we mean the various orders of things arranged in a circle.

Theorem. The number of orders in which n things may be arranged in a circle is (n − 1)!.

Suppose A is at the point where we begin to arrange the digits 1, 2, 3, $\cdots$, n. Suppose we start our arrangement of digits at A with a given digit a. We have then $n-1$ places to fill with the remaining $n-1$ digits. Thus when a is located at A we have $(n-1)!$ arrangements. If we locate a at some other point, such as B, the arrangements which we then get about the point B must be identical with the arrangements which we previously got about the point A. Thus the only *distinct* orders in which the n digits can be arranged in a circle are the $(n-1)!$ permutations we obtained by placing a at A.



PROBLEMS

1. In how many ways may 5 men sit about a circular table?

Solution. Here $n = 5$ and $n - 1 = 4$.

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \text{ ways.}$$

2. In how many ways can 10 men sit about a round table?

3. I have 15 plants. In how many ways can they be planted in a circular bed?

4. In how many ways may 5 ladies and 5 gentlemen be seated at a round dining table so that a lady is seated between each pair of men?

5. In how many ways may 5 men and their wives be seated around the table at dinner so that no man will sit next his wife, but so that the men and women will sit alternately?

6. In how many ways may the letters of the word *Portland* be arranged in a circle?

7. In how many ways can 15 different flags, of which there are 5 red, 5 white, and 5 blue, be arranged in a circle so that there are no two flags of the same color adjacent to each other?

8. In how many ways may 12 different kinds of red flowers and 4 different kinds of white flowers be placed about a circle so that there will be 3 and only 3 red flowers next to each other?

171. Permutations involving duplicates. In the case where we have a number of objects some of which are identical with each other, we make use of the

Theorem. *The number of permutations of n things of which p are alike, taken all together, is* $\frac{n!}{p!}$.

If all the things were different, we should have $n!$ permutations, but since p of the n things are alike, any rearrangement of those p like things among themselves will not affect the permutation. For any fixed arrangement of the n things there are $p!$ different arrangements of the p like things. Thus $\frac{1}{p!}$ of the $n!$ permutations are identical, and there are only $\frac{n!}{p!}$ distinct permutations of the n things p of which are alike.

Corollary. *If of n things p are of one kind, q of another kind, r of another, etc., then there are* $\frac{n!}{p!\,q!\,r!\cdots}$ *permutations of the n things taken all at a time.*

PROBLEMS

1. How many distinct arrangements of the letters of the word *minimum* are possible?

Solution. Here $n = 7$, $p = 3$, and $q = 2$.

Hence there are $\frac{7!}{3!\,2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{2} = 420$ arrangements.

2. How many distinct arrangements of the letters of the word *permutations* are possible?

3. I have 10 signal-flags, 5 red, 2 blue, and 3 white. How many signals may I make if I must fly all the flags at one time?

4. How many signals may I make with the flags of Exercise 3 if the top flag and the bottom flag must both be red?

5. Would 3 dots, 2 long dashes, and 2 short dashes furnish enough material for a telegraphic code to cover the 26 letters, 10 digits, and 6 punctuation marks?

6. There are 5 ways of going down-town, and 10 different errands which I can do when I arrive there. In how many different ways may I go down-town, do one errand, and return?

7. Using the data of Problem 6, find in how many ways I can spend the afternoon if I go down-town, do any or all of the 10 errands, and return.

8. A corporation runs 15 hotels, and has 20 men who are eligible to be managers of these hotels. In how many ways may the company appoint these men to the positions?

9. In how many ways can I run my dining room with 6 waiters, if I have 10 men of whom 2 are able to be head waiter or can fill in as regular waiters?

10. How many five-digit numbers can I form from the numerals 1 to 7, inclusive, if I am allowed to repeat numerals at will?

11. I have 100 books. In how many different ways can I fill a shelf holding 5 books, provided the arrangement on the shelf is of no consequence?

12. I still have the 100 books. If the arrangement on the shelf is considered, in how many ways can I fill the shelf holding 5 books?

13. Six men sit on a straight bench. In how many orders can they arrange themselves?

14. In how many orders can 6 men sit about a round table?

15. I have 100 men on the football squad. Of these, 5 are quarterbacks, 30 play elsewhere in the backfield, 25 are ends, and the remainder play in the line. If there are 4 backs, 2 ends, and 5 men in the line, how many teams may I form from this squad?

172. Probability. The mathematical theory of probability has received wider application than most of the topics considered in this text. The theories of life insurance and of investments, the modern developments of physics, zoölogy, and chemistry, as well as various industries make frequent use of this subject.

In considering the application of the subject of probability it is not within the scope of this chapter to deal with any but the simplest applications of the theory; as, for example, the operation of drawing lots, of throwing dice, or of selecting cards at random from a pack. The only factors which enter into the latter case are the total number of cards and the number of varieties of cards.

Instances of the application of the more advanced theory of probability may be drawn from the field of industry. In a certain telephone exchange, calls are made at the average rate of, let us say, 5000 calls per hour during the day. It is often important to know the probability of two calls' being made within a tenth of a second of each other. The solution of this problem is too involved for this text, although it is perfectly possible to solve it by means of an extension of the methods which will be developed here.

173. Illustration. If a bag contains 3 white balls and 4 black balls, and if 1 ball is taken out at random, what is the chance that the ball drawn will be white?

This question may be answered as follows: There are 7 balls in the bag and we are as likely to get one as another. Thus a ball may be drawn in 7 different ways. Of these 7 possible ways 3 will produce a white ball. Thus the chance that the ball drawn will be white is 3 to 7, or $\frac{3}{7}$. The chance that a black ball will be drawn is $\frac{4}{7}$.

174. General statement. It is plain that we may generalize this illustration as follows: If an event may happen in p ways and fail in q ways, each way being equally probable, the chance or probability that it will happen in one of the p ways is

$$\frac{p}{p+q}. \tag{1}$$

The chance that it will fail is

$$\frac{q}{p+q}. \tag{2}$$

The sum of the chances of the event's happening and failing is 1, as we observe by adding (1) and (2).

The odds in favor of the event are the ratio of the chance of happening to the chance of failure. In this case the odds in favor are

$$\frac{p}{q}. \tag{3}$$

The odds against the event are

$$\frac{q}{p}.$$

PROBLEMS

1. From a pack of 52 cards a single card is drawn. What is the probability that the card is an ace?

Solution. Since there are 52 cards in the pack, the probability that the card drawn will be any given card is $\frac{1}{52}$. There are, however, 4 aces in the pack. This means that there will be 4 times the probability that the card will be one of the 4 aces. The probability is, therefore, $\frac{4}{52}=\frac{1}{13}$.

2. I have three hats and two caps. If I pick up one of them at random, what is the chance that I will pick a cap? a hat? What are the odds in favor of picking a hat?

3. What is the chance of getting a six with a single throw of two dice?

Hint. There are 36 ways two dice can fall. How many of these give a six?

4. A man has a choice of four meats and two desserts. If he chooses one dish at random, what is the probability that it will be a meat? a dessert? What are the odds in favor of the dessert?

5. A man has five suits of clothes consisting of coat, trousers, and vest. If he dresses in the dark, what is the chance that his clothes match?

HINT. How many combinations can he make from the five suits?

175. Probability of successive choices. We have seen (section 167) that if one act may be performed in p ways and a second act may be performed in q ways, the total number of ways in which the two can be performed together is $p \cdot q$. The same reasoning applied to probability leads us to the conclusion that if we have two or more events for which the probability of the first is, let us say, $\frac{3}{4}$, and that of the second event is $\frac{1}{2}$, then the probability that the two events will occur together is the product of their individual probabilities, or $\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$. This fact furnishes us with an illustration of the manner in which the reasoning which we have applied to the subject of permutations and combinations may be extended to include the theory of probability. The fact stated above may be generalized in the following

Theorem. *If in a series of events the individual probabilities are $m, n, r, \cdots$, the probability of all the events' occurring together is $m \cdot n \cdot r \cdots$.*

PROBLEMS

1. What is the probability that I will get three sixes in one throw with three dice?

Solution. Since the chance of getting one six with one throw of one die is $\frac{1}{6}$ we may say that the probability of getting three sixes with three dice is $(\frac{1}{6})(\frac{1}{6})(\frac{1}{6}) = \frac{1}{216}$.

2. A man has four suits of clothes consisting of coat, trousers, and vest. If he dresses at random, what is the chance that he will have on garments that match? What is the chance that the garments will be from only two suits?

3. From a pack of cards two cards are drawn in succession and held. What is the chance that they are both of the same suit?

4. I have a bag containing 6 red, 8 blue, 7 green, and 10 white balls. If I draw three balls in succession, without returning the balls I have drawn, what is the chance that they will all be white?

5. If in Problem 4 the balls are returned after each drawing, what is the chance that all three will be white?

6. I have a bag holding 2 white, 4 black, and 1 red ball. What is the chance of drawing 1 red and 1 black ball in two successive draws if the balls are not returned after each draw? if they are returned?

7. From a pack of 53 cards five cards are drawn. What is the chance that one of the cards is the joker, one an ace, and the other a king of the same suit?

8. What is the probability of an event that is certain to happen? certain not to happen?

9. In throwing two dice, what is the chance that the throw will be six? seven? twelve? ten?

10. What is the most likely number to throw with two dice?

11. I toss a coin five times. What is the likelihood that it will fall heads every time?

12. The chance that an event will happen is $\frac{5}{8}$. What are the odds in its favor? against it?

13. What is the chance of drawing 4 aces from a pack of 52 cards in 4 tries?

14. I toss a coin six times. What is the probability that I will get six tails?

CHAPTER XIX

DETERMINANTS

176. Solution of two linear equations. We have already treated the solution of two linear equations in two variables and given (section 64) a method of solving three or more linear equations in three or more variables. This latter process is rather laborious and can be very much abridged and also developed more symmetrically by the considerations of the present chapter.

Let us solve the equations

$$a_1x + b_1y = c_1, \tag{1}$$

and

$$a_2x + b_2y = c_2. \tag{2}$$

Multiplying (1) by b_2 and (2) by b_1, we obtain

$$a_1b_2x + b_1b_2y = b_2c_1$$

and

$$\underline{a_2b_1x + b_1b_2y = b_1c_2\quad}$$

Subtracting,

$$(a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2,$$

or, if

$$(a_1b_2 - a_2b_1) \neq 0,$$

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \tag{3}$$

and

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \tag{4}$$

We note that the denominators of the expressions for x and y are the same. This denominator we will denote symbolically by the following notation:

$$a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \tag{5}$$

The symbol in the right-hand member is called a **determinant.** Since there are two rows and two columns, this determinant is said to be of the **second order,** and the left-hand member of the equation is called the **development** of the determinant. The symbols a_1, b_1, a_2, and b_2 are called **elements** of the determinant, while the elements a_1 and b_2 are said to comprise its **principal diagonal.**

The numerators in (3) and (4) may also be written as determinants of the second order, thus:

$$b_2c_1 - b_1c_2 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix},$$

and

$$a_1c_2 - a_2c_1 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

Rule. *The development of any determinant of the second order is obtained by subtracting from the product of the elements on the principal diagonal the product of the elements on the other diagonal.*

Thus $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1$; $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2.$

Evidently each term of the development contains one and only one element of each row and each column; that is, for example, the letter x and the subscript 1 appear in each term once and only once.

We can now rewrite the solution (3) and (4) of equations (1) and (2) in determinant form:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}; \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \qquad (6)$$

It is noted that the numerator of the expression for x is formed from the denominator by replacing the column which contains the coefficients of x, a_1 and a_2, by the constant terms c_1 and c_2. Similarly, in the numerator of the expression for y, b_1 and b_2 of the denominator are replaced by c_1 and c_2.

One should keep in mind that the determinant is merely a symbolic form of expression for its development. In the case of determinants of the second order the introduction of the new notation is hardly necessary, as the development itself is simple; just as we would scarcely need to introduce the exponential notation if we had to consider only the squares of numbers. It turns out, however, as we shall see, that we are able to denote by determinants with more than two rows and columns, expressions with which it would be very laborious to operate directly.

177. Solution of three linear equations. Let us solve the equations

$$a_1x + b_1y + c_1z = d_1, \qquad (1)$$

$$a_2x + b_2y + c_2z = d_2, \qquad (2)$$

and

$$a_3x + b_3y + c_3z = d_3. \qquad (3)$$

Eliminating y from (1) and (2) and (1) and (3), we obtain

$$(a_1b_2 - a_2b_1)x + (b_2c_1 - b_1c_2)z = d_1b_2 - d_2b_1,$$

and

$$(a_3b_1 - a_1b_3)x + (b_1c_3 - b_3c_1)z = d_3b_1 - d_1b_3.$$

Eliminating z,

$$[(a_1b_2 - a_2b_1)(c_3b_1 - b_3c_1) - (a_3b_1 - a_1b_3)(b_2c_1 - b_1c_2)]x$$
$$= (d_1b_2 - d_2b_1)(c_3b_1 - b_3c_1) - (d_3b_1 - d_1b_3)(b_2c_1 - b_1c_2).$$

Developing, canceling, and dividing by b_1, we obtain

$$x = \frac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_1b_3c_2 - d_3b_2c_1 - d_2b_1c_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_3b_2c_1 - a_2b_1c_3}. \qquad (4)$$

Following the analogy of the last section, we write the denominator

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_3b_2c_1 - a_2b_1c_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \qquad (5)$$

The right-hand member of this equation we call a determinant of the *third* order, and the left-hand member is its development. As in the determinant of the second order, the elements a_1, b_2, and c_3 comprise the principal diagonal; each term of the development contains one and only one element of each row and each column, and all possible terms so constructed are included in the development. The signs of the terms of the determinant of the third order may be kept in mind by the following device.

Rewrite the first and second columns to the right of the determinant as follows:

$$\left|\begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}\right|.$$

The positive terms are found on the diagonals running down from left to right, the negative terms on the diagonals running up from left to right.

The numerator of the fraction (4) expressed in determinant notation is

$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.$$

We can find similarly the values of y and z that satisfy equations (1), (2), and (3). The complete solution in determinant notation of the equations (1), (2), and (3) is as follows:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}};\; y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}};\; z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}. \qquad (6)$$

The principle observed on page 237 for forming the determinants in the numerators of the expressions for x and y may be followed here. The determinant in the numerator of the expression for x, y, or z is found from that in the denominator by replacing the column that contains the coefficients of the variable in question by a column consisting of constant terms. Thus, to obtain the numerator of the expression for z, we replace the column of coefficients of z as it appears in the denominator by a column composed of the constant terms of the equations. That is, the numerator is just like the denominator except that d_1, d_2, d_3 have replaced c_1, c_2, c_3.

EXERCISES

Find the value of the following determinants:

1. $\begin{vmatrix} 5 & 3 & 1 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{vmatrix}$.

Solution. $\begin{vmatrix} 5 & 3 & 1 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{vmatrix} = 20 + 0 + 2 - 0 - 6 - 30 = -14.$

2. $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix}$.

3. $\begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 2 \\ 5 & 0 & 4 \end{vmatrix}$.

4. $\begin{vmatrix} 5 & 7 & 1 \\ 1 & 0 & 2 \\ 0 & 3 & 1 \end{vmatrix}$.

5. $\begin{vmatrix} 1 & 6 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix}$.

6. $\begin{vmatrix} a & 0 & c \\ c & a & 0 \\ 0 & c & a \end{vmatrix}$.

7. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

8. $\begin{vmatrix} 3 & -1 & 2 \\ -2 & 5 & 0 \\ 0 & 1 & 1 \end{vmatrix}$.

9. $\begin{vmatrix} 3 & 4 & 6 \\ 0 & -1 & 2 \\ 7 & -5 & 0 \end{vmatrix}$.

10. $\begin{vmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 3 & -1 & 0 \end{vmatrix}$.

Solve the following equations using determinants. Check each result.

11. $x + 2y = 0,$
$3x - y = 7.$

Solution. Using the expressions (6), section 176, as formulas, we obtain

$$x = \frac{\begin{vmatrix} 0 & 2 \\ 7 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}} = \frac{0 - 14}{-1 - 6} = 2, \qquad y = \frac{\begin{vmatrix} 1 & 0 \\ 3 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}} = \frac{7 - 0}{-1 - 6} = -1.$$

Check. $2 + 2(-1) = 0;\ 6 - (-1) = 7.$

12. $x = \frac{\begin{matrix}3 & 3\\ 3 & -7\end{matrix}}{\begin{matrix}3 & -7\\ 2 & 3\end{matrix}} = \frac{-21+21}{9+14} = 0$ $\quad y = \frac{\begin{matrix}2 & 3\\ 3 & -1\end{matrix}}{\begin{matrix}3 & -1\\ 2 & 3\end{matrix}} = \frac{9+14}{} = 1$

12. $3x - 7y = -7,$
$2x + 3y = 3.$

13. $x - y = \frac{1}{2},$
$3x + 3y = \frac{5}{2}.$

14. $5x - 9y = -4,$
$2x + y = 3.$

15. $3y + 2x = -6,$
$8x + y = 20.$

16. $x - 3y = -1,$
$3x + y = 7.$

17. $11x - 7y = 27,$
$-3x + 5y = 5.$

18. $2x - 3y + z = 7,$
$x + y = 0,$
$3y - 7z = -17.$

$x = \frac{\begin{matrix}7 & -3 & 1\\ 0 & 1 & 0\\ -17 & 3 & -7\end{matrix}}{\begin{matrix}2 & -3 & 1\\ 1 & 1 & 0\\ 0 & 3 & -7\end{matrix}} = \frac{-49+17}{-14+3-21} = \frac{3}{-3}$

Solution. Rearranging so that the terms in the same variable are in a column, and supplying the zero coefficients, we get

$$\begin{aligned}2x - 3y + 1z &= 7,\\ 1x + 1y + 0z &= 0,\\ 0x + 3y - 7z &= -17.\end{aligned}$$

By (6), section 177,

$$x = \frac{\begin{vmatrix}7 & -3 & 1\\ 0 & 1 & 0\\ -17 & 3 & -7\end{vmatrix}}{\begin{vmatrix}2 & -3 & 1\\ 1 & 1 & 0\\ 0 & 3 & -7\end{vmatrix}} = \frac{-49 + 0 + 0 + 17 - 0 - 0}{-14 + 3 + 0 - 0 - 21 - 0} = \frac{-32}{-32} = 1,$$

$$y = \frac{\begin{vmatrix}2 & 7 & 1\\ 1 & 0 & 0\\ 0 & -17 & -7\end{vmatrix}}{\begin{vmatrix}2 & -3 & 1\\ 1 & 1 & 0\\ 0 & 3 & -7\end{vmatrix}} = \frac{0 + 0 - 17 - 0 - 0 + 49}{-32} = \frac{32}{-32} = -1,$$

$$z = \frac{\begin{vmatrix}2 & -3 & 7\\ 1 & 1 & 0\\ 0 & 3 & -17\end{vmatrix}}{\begin{vmatrix}2 & -3 & 1\\ 1 & 1 & 0\\ 0 & 3 & -7\end{vmatrix}} = \frac{-34 + 21 + 0 - 0 - 0 - 51}{-32} = \frac{-64}{-32} = 2.$$

Check. $2 - 3(-1) + 2 = 7;\ 1 - 1 = 0;\ -3 - 14 = -17.$

19. $2x - 3y = -1,$
$3x + 2z = 5,$
$2y + 3z = 5.$

20. $\frac{1}{5}x - \frac{1}{3}y = 0,$
$\frac{1}{5}x + \frac{1}{2}z = 2,$
$\frac{1}{4}z + \frac{1}{6}y = 1.$

21. $x + y - z = -2,$
$x + z - y = 6,$
$y - x + z = 0.$

22. $3x - 2z = 15,$
$7x + 5y = 74,$
$4y + z = 23.$

23. $x + y - z = -1,$
$2x - 3y + 4z = -6,$
$x - 2y + 4z = 0.$

24. $x + 2y - z = 0,$
$y + 2z - x = \frac{1}{2},$
$z + 2x - y = -\frac{11}{2}.$

25. $x - 2y - 3z = 3,$
$x + 3y - 6z = -2,$
$2y + 4z = -2.$

26. $x + 5y = -17,$
$x - 7z = -51,$
$2y + 3z = 15.$

27. $.1x - .2y + .2z = -2,$
$.3x + .1y - .4z = -4,$
$.5y + .6z = 8.$

28. $ax + by - cz = 2ab,$
$by + cz - ax = 2bc,$
$cz + ax - by = 2ac.$

178. Inversion. In order to find the development of determinants with more than three rows and columns, the idea of an inversion is necessary. If in a series of positive integers a greater integer precedes a lesser, there is said to be an **inversion.** Thus in the series 1 2 3 4 there is no inversion, but in the series 1 2 4 3 there is one inversion, since 4 precedes 3. In 1 4 2 3 there are two inversions, as 4 precedes both 2 and 3; while in 1 4 3 2 there are three inversions, since 4 precedes 2 and 3, and 3 precedes 2.

EXERCISES

How many inversions are there in each of the following series?

1. 4 1 2 3.
2. 3 2 1 4.
3. 1 3 5 2 4.
4. 5 1 4 2 3.
5. 1 4 5 2 3.
6. 2 5 4 1 3.

179. Development of the determinant. In the development of the determinant of order three we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2. \qquad (1)$$

If we keep the order of the letters in each term the same as their order in the principal diagonal (as we have done in the development above), it is observed that the subscripts in the various terms take on all possible permutations of the three digits 1, 2, and 3. The permutations that occur in the positive terms are 123, 231, and 312, in which occur respectively 0, 2, and 2 inversions. The permutations that occur in the subscripts of the negative terms are 321, 213, and 132, in which occur respectively 3, 1, and 1 inversions.

Thus in the subscripts of the positive terms an even number of inversions occur, while in the subscripts of the negative terms an odd number of inversions occur. This means of determining the sign of a term of the development we shall assume in general.

A determinant having n rows and n columns is called a **determinant of the nth order.** When the terms are expressed and arranged after the fashion of (1) the development of such a determinant is defined by the following

Rule. The development of a determinant of the nth order is equal to the algebraic sum of the terms consisting of letters following each other in the same order in which they are found in the principal diagonal but in which the subscripts take on all possible permutations. A term has the positive or negative sign according as there is an even or an odd number of inversions in the subscripts.

This rule for the development of a determinant is useful in practice only when the elements of the determinant are letters with subscripts. When the elements are numbers we shall find the value of the determinant by a more convenient method. At the same time the proof of many properties of determinants can be worked out most conveniently when letters with subscripts are used.

The determinants of order 4 and 5 are

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 \end{vmatrix}.$$

The student may well wonder what right we have to decide on the details of the development of the determinant of the

nth order in so summary a manner. It must be realized that the determinant is not a concept forced upon us by the laws of mathematical operation, as was the case with irrational and imaginary numbers ($\sqrt{2}$ and $\sqrt{-3}$). It is merely a convenient device for denoting in compact form certain algebraic expressions that would otherwise be cumbersome to deal with. The rule for the development of the determinant has been agreed upon by mathematicians as a useful one, and from that scheme of development many properties of this form of notation follow which render the determinant one of the most powerful existing devices for furthering algebraic investigation.

In the rule given above it is assumed that the number of inversions in the subscripts of the principal diagonal is zero. If this number of inversions is not zero, the sign of any term is + or − according as the number of inversions in its subscripts differs from the number in the subscripts of the principal diagonal by an even or an odd number.

Since each term contains every letter $a, b, \cdots, k$, and also every subscript $1, 2, \cdots, n$, one element of each row and column occurs in each term.

EXERCISES

If we consider a fourth-order determinant employing letters a, b, c, d and subscripts 1, 2, 3, 4, what sign should precede each of the following terms of the development?

1. $a_1b_3c_4d_2$. **2.** $a_1b_2c_4d_3$. **3.** $a_4b_2c_1d_3$. **4.** $a_3b_1c_2d_4$.

Do terms of the following form appear in the development of the fourth-order determinant?

5. $a_1b_2c_3d_4$.

6. $a_1c_2b_3d_4$.

7. $d_2c_3b_4d_1$.

8. $d_2a_3b_2c_4$.

9. $a_1c_3b_2c_4$.

10. $a_2c_3b_4a_3$.

Given a fifth-order determinant, using the letters a, b, c, d, e and the subscripts 1, 2, 3, 4, 5, what sign should precede the following terms in the development of such a determinant?

11. $a_5b_4c_1d_3e_2$. **12.** $a_4b_5c_2d_1e_3$. **13.** $a_4b_2c_5d_3e_1$. **14.** $a_1b_4c_3d_5e_2$.

180. Number of terms. We apply the theorem of permutations to prove the following

Theorem. A determinant of the nth order has n! terms in its development.

Since the number of terms is the same as the number of permutations of the n subscripts taken all at a time, the theorem follows immediately from the corollary in section 168.

From this theorem it appears that the determinant of the third order has six terms. This is consistent with the result given on page 242.

The device shown on page 239 for obtaining these terms from the various diagonals of the array is applicable only for the determinant of the third order. If one attempts to obtain the development of the determinant of the fourth order from the diagonals of an array similar to the special arrangement on page 239, only eight terms would be obtained, whereas the theorem just proved calls for twenty-four terms.

181. Multiplication by a constant. In this and the following sections we shall give a number of theorems on determinants which greatly facilitate their evaluation and which make a proof for the solution of any number of linear equations in an equal number of variables a simple matter.

Theorem. If every element of a row or column is multiplied by m, the value of the determinant is multiplied by m.

Suppose that every element of the first row of a determinant is multiplied by m. Since each term of the development contains one and only one element from the first row, every term is multiplied by m; that is, the determinant is multiplied by m.

Illustration.

$$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = ma_1b_2c_3 + ma_2b_3c_1 + ma_3b_1c_2 - ma_3b_2c_1 - ma_2b_1c_3 - ma_1b_3c_2$$

$$= m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Similarly, $\begin{vmatrix} 3 & 2 & 6 \\ 1 & 2 & 4 \\ 5 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 3 \cdot 2 \\ 1 & 2 & 2 \cdot 2 \\ 5 & 1 & 1 \cdot 2 \end{vmatrix} = 2 \begin{vmatrix} 3 & 2 & 3 \\ 1 & 2 & 2 \\ 5 & 1 & 1 \end{vmatrix}.$

182. Interchange of rows and columns. We now prove the following

Theorem. The value of a determinant is not changed if the columns and rows are interchanged.

Take for instance the determinant of order four. We must prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

In each of these determinants the principal diagonal is the same, and hence the developments derived according to the statement on page 243 will be the same for each determinant, since the terms will be identical except for their order. The same reasoning is valid for a determinant of any order.

183. Interchange of rows or columns. We now prove the following

Theorem. If two columns or two rows are interchanged, the sign of the determinant is changed.

Again let us take for example the determinant of order four and fix our attention on the first and second rows. We must prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

In the first determinant the principal diagonal is $a_1b_2c_3d_4$, while in the second the principal diagonal, $a_2b_1c_3d_4$, is obtained from the principal diagonal of the first determinant by one inversion of the subscripts. Hence this term is found among the negative terms of the first determinant.

Since the only difference between the second determinant and the first is the interchange of the subscripts 1 and 2, evidently any term of the second is obtained from some term of the first by a single inversion. Thus if a single inversion is carried out in every term of the first determinant, the various terms of the second are obtained. But since this process changes the sign of every term of the first determinant (page 243), the second determinant must equal the negative of the first. Similar reasoning may be applied to the interchange of any two consecutive rows or columns.

Consider now the effect of interchanging any two rows which are separated, we will say, by k intermediate rows. To bring the lower of the two rows in question to a position next below the upper one by successive interchanges of adjacent rows, we must make k such interchanges. To bring similarly the upper of the two rows to the position previously occupied by the other requires $k + 1$ further interchanges of adjacent rows. Hence the interchange of the two rows is equivalent to $2k + 1$ interchanges of adjacent rows, the effect of which is to change the sign of the determinant, since $2k + 1$ is always an odd number.

184. Identical rows and columns. This leads to the following important

Theorem. *If a determinant has two rows or two columns identical, its value is zero.*

Suppose that the first and the second rows of a determinant are identical. Suppose that the value of the determinant is the number D. By section 183, if we interchange the first and second rows, the value of the resulting determinant is $-D$. But since the interchange of the first two rows, which are *identical*, does not affect the value of the determinant at all, we have

$$D = -D,\ 2D = 0;\ \text{that is,}\ D = 0.$$

Corollary. *If any row (or column) is m times any other row (or column), the value of the determinant is zero.*

By section 181, the determinant may be considered as the product of m and a determinant which has two rows (or columns) identical. Hence this product is equal to zero.

EXERCISES

What is the relation between the pairs of determinants given below? (Find the relationship without evaluating.)

1. $$\begin{vmatrix} 6 & 3 & 2 \\ 4 & 0 & 1 \\ 10 & 1 & 3 \end{vmatrix}, \quad \begin{vmatrix} 3 & 3 & 2 \\ 2 & 0 & 1 \\ 5 & 1 & 3 \end{vmatrix}.$$

2. $$\begin{vmatrix} 1 & 0 & 3 \\ 2 & 2 & 0 \\ 3 & 1 & 3 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 3 \\ 2 & 6 & 0 \\ 3 & 3 & 3 \end{vmatrix}.$$

3. $$\begin{vmatrix} 5 & 0 & 1 \\ 2 & 3 & 1 \\ 4 & 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 2 & 3 & 1 \\ 5 & 0 & 1 \\ 4 & 1 & 2 \end{vmatrix}.$$

4. $$\begin{vmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 4 \\ 3 & 3 & 3 \end{vmatrix}.$$

5. $$\begin{vmatrix} 2 & 1 & 2 \\ 3 & 5 & 4 \\ 1 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 2 \\ 5 & 4 & 3 \\ 0 & 1 & 1 \end{vmatrix}.$$

6. $$\begin{vmatrix} 4 & 0 & 2 \\ 1 & 1 & 0 \\ 3 & 1 & 3 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ 2 & 0 & 3 \\ 4 & 1 & 3 \end{vmatrix}.$$

7. $$\begin{vmatrix} -3 & -1 & -2 \\ 0 & 1 & 3 \\ 5 & 6 & 11 \end{vmatrix}, \quad \begin{vmatrix} 3 & 1 & 2 \\ 0 & 1 & 3 \\ 5 & 6 & 11 \end{vmatrix}.$$

8. $$\begin{vmatrix} -1 & -2 & -3 \\ 2 & 1 & 4 \\ 1 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 2 & 1 & 4 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}.$$

9. $$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \\ 7 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 3 & 0 & 3 \\ 2 & 0 & 3 \\ -7 & -1 & 0 \end{vmatrix}.$$

10. $$\begin{vmatrix} 1 & 3 & 4 \\ 7 & 5 & 9 \\ 8 & 4 & 6 \end{vmatrix}, \quad \begin{vmatrix} -4 & -5 & -3 \\ 6 & 9 & 4 \\ -8 & -7 & -1 \end{vmatrix}.$$

What is the value of the following determinants?

11. $$\begin{vmatrix} 2 & 3 & 5 \\ 2 & 3 & 7 \\ 2 & 3 & 5 \end{vmatrix}.$$

12. $$\begin{vmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 3 \end{vmatrix}.$$

13. $$\begin{vmatrix} 2 & 0 & 3 \\ 8 & 7 & 6 \\ 2 & 0 & 3 \end{vmatrix}.$$

14. $$\begin{vmatrix} 5 & 3 & 1 \\ 7 & 8 & 9 \\ 10 & 6 & 2 \end{vmatrix}.$$

15. $$\begin{vmatrix} 0 & 0 & 0 & 0 \\ 3 & 8 & 7 & 2 \\ 7 & 5 & 1 & 0 \\ 2 & 1 & 1 & 4 \end{vmatrix}.$$

16. $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \end{vmatrix}.$$

17. $$\begin{vmatrix} 1 & 0 & 3 & 9 \\ 2 & 2 & 6 & 0 \\ 4 & 3 & 12 & 3 \\ 6 & 7 & 18 & 1 \\ 8 & 1 & 24 & 10 \end{vmatrix}.$$

185. Development by minors. In the development of the determinant of the third order, page 238, we may combine the terms as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}. \qquad (1)$$

We observe that the coefficient of a_1 is the determinant that we obtain by erasing the row and the column in which a_1 lies. A similar fact holds for the coefficients of a_2 and a_3.

The determinant obtained by erasing the row and the column in which a given element lies is called the **minor** of that element. Thus $\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$ is the minor of a_2.

We notice that in the above development by minors (1) the sign of a given term is a + or a −, according as the sum of the number of the row and the number of the column of the element in that term is even or odd. Thus in the first term a_1 is in the first row and the first column, and since $1 + 1 = 2$, the statement is verified for that case. Similarly, a_2 is in the first column and the second row, and since $1 + 2 = 3$, an odd number, the sign is minus and the law holds here. The last term is positive, which we would expect since a_3 is in the first column and the third row, and $1 + 3 = 4$. The proof for the general validity of this law of signs follows in the next section.

The elements of any row or column other than the first may be taken and the development given in terms of the minors with respect to such elements. For instance, take the development with respect to the elements of the second row,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}.$$

The rule of signs is the same as given above; that is, for instance, the last term is negative, as c_2 is in the third column and the second row, and $2+3=5$. By generalizing these considerations we may find the development of a determinant by minors by the following

Rule. *Write in succession the elements of any row or column, each multiplied by its minor.*

Give each term a + or a − sign, according as the sum of the number of the row and the number of the column of the element in that term is even or odd.

Develop the determinant in each term by a similar process until the value of the development can be determined directly by multiplication.

EXAMPLE

Develop the following determinant by minors:

$$\begin{vmatrix} 3 & 4 & 0 & 9 \\ 7 & 0 & 1 & 4 \\ 9 & 1 & 2 & 6 \\ 10 & 2 & 1 & 5 \end{vmatrix}.$$

Solution. When a determinant is to be developed by minors, it is often a considerable saving in labor to select the column or row which has the greatest number of zeros and 1's. In the present case the third column seems to satisfy these conditions best.

Developing by minors using the elements of the third column,

$$\begin{aligned}
\begin{vmatrix} 3 & 4 & 0 & 9 \\ 7 & 0 & 1 & 4 \\ 9 & 1 & 2 & 6 \\ 10 & 2 & 1 & 5 \end{vmatrix}
&= +0\begin{vmatrix} 7 & 0 & 4 \\ 9 & 1 & 6 \\ 10 & 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 3 & 4 & 9 \\ 9 & 1 & 6 \\ 10 & 2 & 5 \end{vmatrix} \\
&\quad + 2\begin{vmatrix} 3 & 4 & 9 \\ 7 & 0 & 4 \\ 10 & 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 3 & 4 & 9 \\ 7 & 0 & 4 \\ 9 & 1 & 6 \end{vmatrix} \\
&= 0 - \left(-4\begin{vmatrix} 9 & 6 \\ 10 & 5 \end{vmatrix} + 1\begin{vmatrix} 3 & 9 \\ 10 & 5 \end{vmatrix} - 2\begin{vmatrix} 3 & 9 \\ 9 & 6 \end{vmatrix}\right) \\
&\quad + 2\left(-4\begin{vmatrix} 7 & 4 \\ 10 & 5 \end{vmatrix} + 0 - 2\begin{vmatrix} 3 & 9 \\ 7 & 4 \end{vmatrix}\right) \\
&\quad - \left(-4\begin{vmatrix} 7 & 4 \\ 9 & 6 \end{vmatrix} + 0 - 1\begin{vmatrix} 3 & 9 \\ 7 & 4 \end{vmatrix}\right) \\
&= [4(-15) + 75 - 126] + 2(20 + 102) \\
&\quad - (-24 + 51) \\
&= -60 + 75 - 126 + 40 + 204 + 24 - 51 \\
&= 106.
\end{aligned}$$

186. Proof for development by minors. On referring to the rule of section 185 we observe that in order to show that the development by minors is the same as the development obtained by the definition on page 243, we must prove the two following statements.

I. The coefficient of any element in the development of a determinant is the minor of that element (apart from the sign).

II. Each element times its minor should have a + or a − sign according as the sum of the number of the row and the column of the element is even or odd.

Consider the element a_1.

Each term that contains a_1 must contain every letter other than a, and the subscripts of these letters must take on all permutations of the numbers $2, 3, \cdots, n$. This coefficient of a_1 contains, then, by section 179, all the terms of its minor.

Since a_1 is in the first place in each term, the only inversions in the subscripts are those among the numbers $2, 3, \cdots, n$. Hence the sign of each term in the coefficient of a_1 is positive or negative according to the number of inversions in its subscripts. Hence the coefficient of a_1 in the development of the determinant by the rule of section 179 is exactly its minor, and our theorem is established for the element a_1.

Consider now any element, as d_5, which occurs in the fifth row and the fourth column. Interchange adjacent rows and columns until d_5 is brought into the leading position in the principal diagonal. This requires, in all, seven interchanges, three to get the d_5 in the first column, and then four to get it into the first row. This changes the sign of the determinant seven times, leaving it the negative of its original value. By the reasoning just given in the case of a_1, the coefficient of d_5 (now in the position first held by a_1) in the original determinant would be the minor of d_5, except that the signs would all be changed. Hence the term consisting of d_5 times its minor has the − sign, and the theorem is proved for this case.

In general, to bring a term in the ith row and the kth column to the leading position requires $i-1+k-1=i+k-2$ interchanges of adjacent rows or columns. This involves $i+k-2$, or what amounts to the same thing, $i+k$ changes of sign. Hence a positive or a negative sign should be given to the element times its minor according as the sum $i+k$ is even or odd.

EXERCISES

Find the value of the following determinants, using the development of minors. (See Example, p. 250.)

1. $$\begin{vmatrix} 2 & 1 & 4 \\ 3 & 0 & 5 \\ 1 & 2 & 3 \end{vmatrix}.$$

2. $$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 6 & 0 & 7 \end{vmatrix}.$$

3. $$\begin{vmatrix} 0 & 1 & 3 \\ 2 & 0 & 7 \\ 3 & 1 & 0 \end{vmatrix}.$$

4. $$\begin{vmatrix} 3 & -1 & 1 \\ 2 & 0 & -3 \\ 5 & -2 & -1 \end{vmatrix}.$$

5. $$\begin{vmatrix} a & b & 0 \\ a & 0 & c \\ 0 & b & c \end{vmatrix}.$$

6. $$\begin{vmatrix} a & 0 & b \\ c & d & 0 \\ 0 & e & f \end{vmatrix}.$$

7. $$\begin{vmatrix} x & 0 & y \\ 0 & z & 0 \\ m & 0 & p \end{vmatrix}.$$

8. $$\begin{vmatrix} 3 & 2 & 1 & 0 \\ 5 & 0 & 1 & 0 \\ 3 & 2 & 0 & 1 \\ 4 & 3 & 1 & 0 \end{vmatrix}.$$

9. $$\begin{vmatrix} 7 & 9 & 1 & 3 \\ 4 & 0 & 7 & 2 \\ 3 & 0 & 8 & 1 \\ 2 & 0 & 7 & 6 \end{vmatrix}.$$

10. $$\begin{vmatrix} 2 & 8 & 1 & 9 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 7 & 5 & 3 & 1 \end{vmatrix}.$$

11. $$\begin{vmatrix} 3 & 0 & -1 & 2 \\ 4 & 1 & 1 & 1 \\ 7 & 0 & 2 & 5 \\ 3 & 1 & 0 & 2 \end{vmatrix}.$$

12. $$\begin{vmatrix} x & 0 & 0 & y \\ 0 & 0 & x & y \\ 0 & x & y & 0 \\ x & y & 0 & 0 \end{vmatrix}.$$

13. $$\begin{vmatrix} 3 & 7 & 0 & 1 \\ 3 & 7 & 3 & 4 \\ 3 & 7 & 2 & -1 \\ 3 & 7 & -3 & 0 \end{vmatrix}.$$

14. $$\begin{vmatrix} 3 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 3 & 0 & 3 & 4 \\ 0 & 1 & 0 & 7 \end{vmatrix}.$$

15. $$\begin{vmatrix} 8 & 0 & 1 & 7 \\ -2 & 1 & 0 & 3 \\ 5 & -6 & -2 & -4 \\ 8 & -1 & 1 & -1 \end{vmatrix}.$$

16. $$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 2 & 1 & 0 & 3 \end{vmatrix}.$$

17. $$\begin{vmatrix} a & b & c & 0 \\ a & d & e & f \\ 0 & a & 0 & 0 \\ a & 0 & b & 0 \end{vmatrix}.$$

18. $$\begin{vmatrix} x & y & 0 & 0 & 0 \\ 0 & x & y & 0 & 0 \\ 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & x & y \\ y & 0 & 0 & 0 & x \end{vmatrix}.$$

19. $$\begin{vmatrix} 1 & 3 & 0 & 2 & 1 \\ 8 & 7 & 5 & 0 & 1 \\ 3 & 2 & 7 & 8 & 0 \\ 4 & 0 & 3 & 9 & 1 \\ 8 & 7 & 3 & 6 & 2 \end{vmatrix}.$$

20. $$\begin{vmatrix} 5 & 3 & 1 & -1 & -3 \\ 2 & 0 & 0 & 0 & -1 \\ 8 & 7 & 9 & 3 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & 3 & 3 \end{vmatrix}.$$

187. Sum of determinants. We now prove the fact that under certain conditions the sum of two determinants may be written in determinant form. The fact that the product of two determinants is always a determinant is extremely important for certain more advanced topics in mathematics, but the proof lies beyond the scope of this chapter.

Theorem. If each of the elements of any row or column consists of the sum of two numbers, the determinant may be written as the sum of two determinants.

For example, we shall prove that

$$\begin{vmatrix} a_1 + a'_1 & b_1 & c_1 \\ a_2 + a'_2 & b_2 & c_2 \\ a_3 + a'_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a'_1 & b_1 & c_1 \\ a'_2 & b_2 & c_2 \\ a'_3 & b_3 & c_3 \end{vmatrix}.$$

Develop the first determinant by minors with respect to the first column, where we symbolize the minors of $a_1 + a'_1$, $a_2 + a'_2$, $a_3 + a'_3$ by A_1, A_2, A_3, respectively.

We have

$$\begin{aligned} \begin{vmatrix} a_1 + a'_1 & b_1 & c_1 \\ a_2 + a'_2 & b_2 & c_2 \\ a_3 + a'_3 & b_3 & c_3 \end{vmatrix} &= (a_1 + a'_1)A_1 - (a_2 + a'_2)A_2 + (a_3 + a'_3)A_3 \\ &= a_1A_1 - a_2A_2 + a_3A_3 + a'_1A_1 - a'_2A_2 + a'_3A_3 \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a'_1 & b_1 & c_1 \\ a'_2 & b_2 & c_2 \\ a'_3 & b_3 & c_3 \end{vmatrix}. \end{aligned}$$

The method of proof given is applicable to the case of any row or column of a determinant of order n.

188. Vanishing of a determinant. For the solution of systems of linear equations we shall make use of the

Theorem. If in the development of a determinant in terms of the minors with respect to a certain column (or row) the elements of that column (or row) are replaced by another column (or row), the resulting development equals zero.

For example, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4, \tag{1}$$

where A_1 represents the minor of a_1. We have to prove that if we replace the a's, for example, by the b's, the result equals zero; that is, that

$$b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4 = 0. \tag{2}$$

This expression (2), when written in determinant form, evidently would have the same form as the left-hand member of (1), except that the first column would consist of b_1, b_2, b_3, and b_4. We should then have two identical columns of the determinant, which would equal zero (section 184). Thus the development in (2) vanishes identically. This method of proof is valid for the general case.

Corollary. *The value of the determinant is unchanged if the elements of any row (or column) are replaced by the elements of that row (or column) increased by a multiple of the elements of another row (or column).*

The proof follows directly from sections 184, 187, and the preceding theorem.

Thus, for example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+nb_1 & b_1 & c_1 \\ a_2+nb_2 & b_2 & c_2 \\ a_3+nb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} nb_1 & b_1 & c_1 \\ nb_2 & b_2 & c_2 \\ nb_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + n\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

189. Practical directions for evaluating determinants. In finding the value of a numeric determinant, the object is to form one in which as many as possible of the elements of some row or column are zero. One should ask oneself the following questions before attempting to evaluate a determinant:

I. Is any row (or column) equal to any other row (or column)? If so, apply section 184.

II. Are the elements of any row (or column) multiples of any other row (or column)? If so, apply section 184.

III. If we add (or subtract) a multiple of the elements of one row (or column) to the elements of another, will an element become zero? If so, apply section 185.

EXERCISES

Evaluate the following determinants:

1. $\begin{vmatrix} 3 & 8 & 4 & 7 \\ 2 & 5 & 3 & 4 \\ 0 & 2 & 1 & 1 \\ 1 & 3 & 2 & 2 \end{vmatrix}.$

Solution. Subtracting the first column from the third, and the fourth column from the second,

$$\begin{vmatrix} 3 & 1 & 1 & 7 \\ 2 & 1 & 1 & 4 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = 0.$$

2. $\begin{vmatrix} 8 & 4 & 6 & 4 \\ 6 & 1 & 1 & 3 \\ 4 & 5 & 6 & 2 \\ 3 & 2 & 0 & 1 \end{vmatrix}.$

Solution. Subtracting twice the fourth column from the first column, and developing,

$$\begin{vmatrix} 8 & 4 & 6 & 4 \\ 6 & 1 & 1 & 3 \\ 4 & 5 & 6 & 2 \\ 3 & 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 6 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 5 & 6 & 2 \\ 1 & 2 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 4 & 6 & 4 \\ 1 & 1 & 3 \\ 5 & 6 & 2 \end{vmatrix}.$$

Subtracting the first column from the second column,

$$-\begin{vmatrix} 4 & 6 & 4 \\ 1 & 1 & 3 \\ 5 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 4 & 2 & 4 \\ 1 & 0 & 3 \\ 5 & 1 & 2 \end{vmatrix}.$$

Subtracting one half the first row from the third row, and developing,

$$-\begin{vmatrix}4&2&4\\1&0&3\\5&1&2\end{vmatrix} = -\begin{vmatrix}4&2&4\\1&0&3\\3&0&0\end{vmatrix} = -\left(-2\begin{vmatrix}1&3\\3&0\end{vmatrix}\right) = 2(-9) = -18.$$

3. $\begin{vmatrix}2&3&7&5\\1&0&2&1\\3&3&8&6\\3&7&2&8\end{vmatrix}.$

4. $\begin{vmatrix}1&0&0&1\\3&0&4&2\\8&7&0&3\\2&5&3&0\end{vmatrix}.$

5. $\begin{vmatrix}1&0&0&1&1\\0&0&1&1&1\\0&1&1&1&0\\1&1&1&0&0\\1&1&0&0&1\end{vmatrix}.$

6. $\begin{vmatrix}0&1&1&1&1\\1&0&1&1&1\\1&1&0&1&1\\1&1&1&0&1\\1&1&1&1&0\end{vmatrix}.$

7. $\begin{vmatrix}3&6&8&4&2\\8&9&6&3&1\\7&0&3&0&1\\2&9&3&1&5\\0&1&0&2&0\end{vmatrix}.$

8. $\begin{vmatrix}3&x&2&0&1\\7&5&4&x&1\\2&3&0&5&x\\3&1&x&2&0\\x&5&3&0&0\end{vmatrix}.$

9. $\begin{vmatrix}4&7&3&2&0\\0&1&1&0&1\\3&8&2&0&1\\8&14&6&4&0\\1&0&0&1&0\end{vmatrix}.$

10. $\begin{vmatrix}25&30&5&0\\15&18&3&0\\8&10&3&7\\4&0&1&2\end{vmatrix}.$

11. $\begin{vmatrix}3&2&1&8&5\\4&10&13&6&9\\0&1&3&2&0\\8&9&3&7&2\\0&1&0&3&0\end{vmatrix}.$

12. $\begin{vmatrix}2&4&6&8&10\\3&5&7&10&12\\0&0&1&0&3\\2&0&1&2&3\\3&4&6&9&13\end{vmatrix}.$

13. $\begin{vmatrix}a&b&a&c\\a&b&d&e\\a&b&c&b\\a&b&f&a\end{vmatrix}.$

14. $\begin{vmatrix}a&a&a&a\\a&b&b&b\\a&b&c&c\\a&b&c&d\end{vmatrix}.$

15. $\begin{vmatrix}a&b&c&d\\a&c&d&b\\a&d&b&c\\a&c&b&d\end{vmatrix}.$

16. $\begin{vmatrix}a&3&b&4\\3&b&4&a\\b&4&a&3\\4&a&3&b\end{vmatrix}.$

190. Solution of linear equations. Suppose that we have given n linear equations in n variables. We seek the form of the solution of the equations in terms of determinants. For simplicity let $n = 4$. Given

$$a_1x + b_1y + c_1z + d_1w = f_1, \tag{1}$$

$$a_2x + b_2y + c_2z + d_2w = f_2, \tag{2}$$

$$a_3x + b_3y + c_3z + d_3w = f_3, \tag{3}$$

and

$$a_4x + b_4y + c_4z + d_4w = f_4. \tag{4}$$

The coefficients of the variables taken in the order in which they are written may be taken as forming a determinant D, which we call the determinant of the system. Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = D.$$

We assume for the present that $D \neq 0$.

Symbolize by A_1, A_2, etc. the minors of a_1, a_2, etc. in this determinant. Multiply (1), (2), (3), (4), respectively, by $A_1, -A_2, A_3, -A_4$. We obtain

$$A_1a_1x + A_1b_1y + A_1c_1z + A_1d_1w = A_1f_1,$$

$$-A_2a_2x - A_2b_2y - A_2c_2z - A_2d_2w = -A_2f_2,$$

$$A_3a_3x + A_3b_3y + A_3c_3z + A_3d_3w = A_3f_3,$$

and

$$-A_4a_4x - A_4b_4y - A_4c_4z - A_4d_4w = -A_4f_4.$$

If we add these equations, the coefficient of x is the determinant D, while the coefficients of y, z, and w are zero (section 188). The right-hand member of the equation is the determinant D except that the elements of the first column are replaced by f_1, f_2, f_3, and f_4 respectively. Hence

$$x = \frac{\begin{vmatrix} f_1 & b_1 & c_1 & d_1 \\ f_2 & b_2 & c_2 & d_2 \\ f_3 & b_3 & c_3 & d_3 \\ f_4 & b_4 & c_4 & d_4 \end{vmatrix}}{D}.$$

In a similar manner we may obtain, as values of y, z, and w, the following:

$$y = \frac{\begin{vmatrix} a_1 & f_1 & c_1 & d_1 \\ a_2 & f_2 & c_2 & d_2 \\ a_3 & f_3 & c_3 & d_3 \\ a_4 & f_4 & c_4 & d_4 \end{vmatrix}}{D};\ z = \frac{\begin{vmatrix} a_1 & b_1 & f_1 & d_1 \\ a_2 & b_2 & f_2 & d_2 \\ a_3 & b_3 & f_3 & d_3 \\ a_4 & b_4 & f_4 & d_4 \end{vmatrix}}{D};\ w = \frac{\begin{vmatrix} a_1 & b_1 & c_1 & f_1 \\ a_2 & b_2 & c_2 & f_2 \\ a_3 & b_3 & c_3 & f_3 \\ a_4 & b_4 & c_4 & f_4 \end{vmatrix}}{D}.$$

It must be observed that we have tacitly assumed in the very writing down of the above equations, (1), (2), (3), and (4), that there is a set of values that satisfy them. The foregoing procedure merely shows that any set of values of x, y, z, and w which satisfy the equations must necessarily have the forms derived. In order to verify the fact that these values do actually satisfy the equations it is necessary to substitute them for the variables, in the same manner that the work is usually checked. This process is virtually reversing the order of the procedure by which the results were derived, and will not be performed here. In each exercise the work should be carefully checked.

When a system of equations has a solution the equations are said to be consistent.

EXAMPLE

Solve for x:

$$\begin{aligned} x + y - z + w &= 0, \\ 2x + 2z - w &= 6, \\ 3y + z + 2w &= 3, \\ 3x - y + z + w &= 0. \end{aligned}$$

Solution. Supplying zero coefficients,

$$\begin{aligned} 1x + 1y - 1z + 1w &= 0, \\ 2x + 0y + 2z - 1w &= 6, \\ 0x + 3y + 1z + 2w &= 3, \\ 3x - 1y + 1z + 1w &= 0. \end{aligned}$$

$$D=\begin{vmatrix}1 & 1 & -1 & 1\\ 2 & 0 & 2 & -1\\ 0 & 3 & 1 & 2\\ 3 & -1 & 1 & 1\end{vmatrix}=\begin{vmatrix}1 & 0 & 0 & 0\\ 2 & 2 & 1 & -3\\ 0 & 4 & 3 & 2\\ 3 & 0 & 2 & -2\end{vmatrix}=\begin{vmatrix}2 & 1 & -3\\ 4 & 3 & 2\\ 0 & 2 & -2\end{vmatrix}$$

$$=\begin{vmatrix}2 & 1 & -2\\ 4 & 3 & 5\\ 0 & 2 & 0\end{vmatrix}=-2\begin{vmatrix}2 & -2\\ 4 & 5\end{vmatrix}=-36.$$

$$x=\frac{\begin{vmatrix}0 & 1 & -1 & 1\\ 6 & 0 & 2 & -1\\ 3 & 3 & 1 & 2\\ 0 & -1 & 1 & 1\end{vmatrix}}{-36}=\frac{\begin{vmatrix}0 & 2 & -2 & 1\\ 6 & -1 & 3 & -1\\ 3 & 5 & -1 & 2\\ 0 & 0 & 0 & 1\end{vmatrix}}{-36}=\frac{\begin{vmatrix}0 & 2 & -2\\ 6 & -1 & 3\\ 3 & 5 & -1\end{vmatrix}}{-36}$$

$$=\frac{\begin{vmatrix}0 & 2 & 0\\ 6 & -1 & 2\\ 3 & 5 & 4\end{vmatrix}}{-36}=\frac{-2\begin{vmatrix}6 & 2\\ 3 & 4\end{vmatrix}}{-36}=\frac{-2(24-6)}{-36}=\frac{-36}{-36}=1.$$

When $D=0$, we cannot find a solution for the equations unless all the numerators in the foregoing expressions also vanish. This case is considered in the next section.

191. Systems whose determinants equal zero. If the determinant of any system of equations vanishes, there is no solution for the system unless all the numerators of the expressions for the variables (as given on page 257) also vanish, thus affording apparent solutions of the form $\frac{0}{0}$.

As an illustration let us consider the system

$$\begin{aligned}x+3y&=2,\\ 2x+6y&=4.\end{aligned}\qquad(1)$$

Here $D=\begin{vmatrix}1 & 3\\ 2 & 6\end{vmatrix}$, and in the formal solutions the numerators for both x and y are zero. Thus $x=\frac{0}{0}$ and $y=\frac{0}{0}$. An inspection of the equations shows, however, that there is not merely one set of values that satisfy the two equations, but that apart from a constant factor, 2, the equations are identical, and hence every pair of values that satisfies one of them satisfies the other also. If, however, the constant term of the second equation had been 3 instead of 4, there would have been no values that satisfied both equations.

A complete discussion of the situation when $D = 0$ is too complex for consideration here. One case, however, where we have three or four equations in three or four variables, when at least one minor of the determinant of the system does not vanish, will be considered.

Take

$$a_1x + b_1y + c_1z + d_1w = f_1, \tag{2}$$

$$a_2x + b_2y + c_2z + d_2w = f_2, \tag{3}$$

$$a_3x + b_3y + c_3z + d_3w = f_3, \tag{4}$$

and

$$a_4x + b_4y + c_4z + d_4w = f_4, \tag{5}$$

where $D = 0$, $A_1 \neq 0$, and the numerator of the expression for each of the variables vanishes.

The minor A_1 is the determinant of the system

$$\begin{aligned} b_2y + c_2z + d_2w &= f_2 - a_2x, \\ b_3y + c_3z + d_3w &= f_3 - a_3x, \\ b_4y + c_4z + d_4w &= f_4 - a_4x. \end{aligned}$$

Hence, if this system has any solution, the value of y must be

$$y = \frac{\begin{vmatrix} (f_2 - a_2x) & c_2 & d_2 \\ (f_3 - a_3x) & c_3 & d_3 \\ (f_4 - a_4x) & c_4 & d_4 \end{vmatrix}}{\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}} = \frac{\begin{vmatrix} f_2 & c_2 & d_2 \\ f_3 & c_3 & d_3 \\ f_4 & c_4 & d_4 \end{vmatrix} - x \begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix}}{A_1}, \tag{6}$$

where x may have any arbitrary value. An inspection of (6) shows us, at once, that since the expression for y involves the variable x there are as many values of y as there are of x. In other words, for each value of x which we substitute in the expression for y, we will have a separate and distinct value of y. This is the same situation we found in equations (1).

In order to obtain one set of values it is convenient to let $x = 0$. We have, then, as values for y, z, and w which will satisfy equations (3), (4), and (5), the following values:

$$y = \frac{\begin{vmatrix} f_2 & c_2 & d_2 \\ f_3 & c_3 & d_3 \\ f_4 & c_4 & d_4 \end{vmatrix}}{A_1}; \quad z = \frac{\begin{vmatrix} b_2 & f_2 & d_2 \\ b_3 & f_3 & d_3 \\ b_4 & f_4 & d_4 \end{vmatrix}}{A_1}; \quad w = \frac{\begin{vmatrix} b_2 & c_2 & f_2 \\ b_3 & c_3 & f_3 \\ b_4 & c_4 & f_4 \end{vmatrix}}{A_1}.$$

That these values also satisfy (2) appears from the fact that, when substituted for the values of y, z, and w in (2), the result is the numerator of the value of x in section 190, which is zero under our present hypothesis.

As a practical matter, if one must find solutions of systems of equations whose determinant is zero, one may proceed as follows:

If not only the determinant of the system but also all the numerators of the expressions for the variables in section 190 vanish, there may be solutions for the system.

If any minor of any element in the determinant is not zero, solve the system in $n - 1$ variables of which this minor is the system, allowing the remaining variable to take any convenient value, usually zero.

By allowing the remaining variable to take on various values, as many sets of values satisfying the system may be found as are desired.

EXERCISES

Find a solution of the following systems by determinants:

1. $2x + y - z - w = 6,$
   $x - 2y + 2z + 3w = -7,$
   $3x - 2z + w = -3,$
   $3x - y + z + 2w = -1.$

Solution.

$$D = \begin{vmatrix} 2 & 1 & -1 & -1 \\ 1 & -2 & 2 & 3 \\ 3 & 0 & -2 & 1 \\ 3 & -1 & 1 & 2 \end{vmatrix} = 0.$$

All remaining fourth-order determinants of this system are zero. We must therefore solve the following system as shown in section 191.

$$\begin{aligned} -2y+2z+3w&=-7-x,\\ +0y-2z+1w&=-3-3x,\\ -1y+1z+2w&=-1-3x. \end{aligned}$$

Whence

$$x=0;\quad y=\frac{\begin{vmatrix}-7 & 2 & 3\\ -3 & -2 & 1\\ -1 & 1 & 2\end{vmatrix}}{\begin{vmatrix}-2 & 2 & 3\\ 0 & -2 & 1\\ -1 & 1 & 2\end{vmatrix}}=\frac{-\begin{vmatrix}-10 & 4\\ -5 & 5\end{vmatrix}}{-2\begin{vmatrix}-2 & 3\\ -1 & 2\end{vmatrix}}=15;$$

$$z=\frac{\begin{vmatrix}-2 & -7 & 3\\ 0 & -3 & 1\\ -1 & -1 & 2\end{vmatrix}}{2}=\frac{-\begin{vmatrix}-5 & -1\\ -3 & 1\end{vmatrix}}{2}=4;$$

$$w=\frac{\begin{vmatrix}-2 & 2 & -7\\ 0 & -2 & -3\\ -1 & 1 & -1\end{vmatrix}}{2}=\frac{-\begin{vmatrix}0 & -5\\ -2 & -3\end{vmatrix}}{2}=5.$$

2. $$\begin{aligned} 2x+3y-z&=-3,\\ x-2y+z&=6,\\ 3x+y&=3. \end{aligned}$$

3. $$\begin{aligned} 5x+2y-z&=8,\\ 3x+z&=0,\\ 2x+2y-2z&=8. \end{aligned}$$

4. $$\begin{aligned} x+y+z&=2,\\ x-y-z&=0,\\ 2y+2z&=2. \end{aligned}$$

5. $$\begin{aligned} 11y+3z&=8,\\ 2x-5y&=-5,\\ 2x+6y+3z&=3. \end{aligned}$$

6. $$\begin{aligned} 3x-5y+2z-6w&=-7,\\ x+2y+7w&=14,\\ 3y+8w&=21,\\ x+5y+15w&=35. \end{aligned}$$

192. Solution of homogeneous linear equations. The equations considered in section 190 become homogeneous (p. 147) if $f_1=f_2=f_3=f_4=0$. We have then

$$\begin{aligned} a_1x+b_1y+c_1z+d_1w&=0,\\ a_2x+b_2y+c_2z+d_2w&=0,\\ a_3x+b_3y+c_3z+d_3w&=0,\\ a_4x+b_4y+c_4z+d_4w&=0. \end{aligned}\tag{I}$$

These equations have evidently the solution

$$x = y = z = w = 0.$$

This we call the zero solution. If we carry out the method of section 191, we observe that the determinant equals zero in the numerator of every fraction giving the value of one of the variables, since all the f's are equal to zero. Thus if D is not equal to zero, the only solution of the above equations is the zero solution. This gives us the following

Rule. *A system of n linear homogeneous equations in n variables has a solution distinct from the zero solution only when the determinant of the system vanishes.*

We seek the condition that the coefficients must fulfill in order that other solutions may also exist. Whether a solution distinct from the zero solution *always* exists when the determinant of the system equals zero we shall not determine, as a complete discussion of this question would be beyond the scope of this chapter. If, however, any solution apart from the zero solution exists, there are countless other solutions. This follows from the following

Theorem. *If x_1, y_1, z_1, w_1 is a solution of equation* (I) *and k is any number, then kx_1, ky_1, kz_1, kw_1 is also a solution.*

The proof for this theorem is evident on substituting kx_1 etc. in equations (I) and observing that the number k is a factor of each equation. Thus if a system of n linear homogeneous equations has any solution distinct from the zero solution it has an infinite number of solutions.

If the determinant of this system equals zero, but if there is any minor of the determinant which is not zero, following the method given in section 191 the system in $n - 1$ variables of which this minor is the determinant may be solved, giving the remaining variable any convenient arbitrary value.

EXAMPLE

Solve by determinants:

$$
\begin{aligned}
x + 3y - 2z + 4w &= 0,\\
2x + y + z - 2w &= 0,\\
y + 3z - w &= 0,\\
3x + 4y - z + 2w &= 0.
\end{aligned}
$$

Solution.

$$D = \begin{vmatrix} 1 & 3 & -2 & 4 \\ 2 & 1 & 1 & -2 \\ 0 & 1 & 3 & -1 \\ 3 & 4 & -1 & 2 \end{vmatrix} = 0.$$

Following the method of section 191,

$$
\begin{aligned}
1y + 1z - 2w &= -2x,\\
1y + 3z - 1w &= 0,
\end{aligned}
$$

and

$$4y - 1z + 2w = -3x.$$

$$y = \frac{\begin{vmatrix} -2x & 1 & -2 \\ 0 & 3 & -1 \\ -3x & -1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -2 \\ 1 & 3 & -1 \\ 4 & -1 & 2 \end{vmatrix}} = \frac{x\begin{vmatrix} -2 & 1 & -2 \\ 0 & 3 & -1 \\ -3 & -1 & 2 \end{vmatrix}}{5\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix}} = \frac{-5x\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix}}{5\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix}} = -x;$$

$$z = \frac{x\begin{vmatrix} 1 & -2 & -2 \\ 1 & 0 & -1 \\ 4 & -3 & 2 \end{vmatrix}}{5\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix}} = \frac{-x\begin{vmatrix} -2 & -1 \\ -3 & 6 \end{vmatrix}}{5\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix}} = \frac{15x}{25} = \frac{3x}{5};$$

$$w = \frac{x\begin{vmatrix} 1 & 1 & -2 \\ 1 & 3 & 0 \\ 4 & -1 & -3 \end{vmatrix}}{25} = \frac{20x}{25} = \frac{4x}{5}.$$

A convenient number to substitute for x is 5. Thus we find that when $x = 5$, $y = -5$, $z = 3$, and $w = 4$.

EXERCISES

Use determinants to solve the following linear systems:

1. $x + y + 2z = 0,$
   $2x + 3y + 3z = 0,$
   $2x + 4y + 2z = 0.$

2. $x + 2y + 3z = 13,$
   $2x + y + z = 7,$
   $3x + 4y + 3z = 21.$

3. $2x + 4y - 3z = 3,$
$3x - 8y + 6z = 1,$
$5x - 4y + 3z = 4.$

4. $3x + y + w = 2,$
$6x + 3z + w = 4,$
$x + 4y + 3z = 3,$
$8y + 5z + 3w = 5.$

5. $3x - 4y + 2z = 1,$
$5x - 5y + 4z = 7,$
$2x - y + 2z = 6.$

6. $x + 2y + z = 1,$
$2x - y - 2z = -9,$
$3x - y + 4z = 3.$

7. $2x - 3y + 4z = 1,$
$3y + 2z - 3w = 2,$
$-z - 7w + 2x = -1,$
$w - 2x + 3y = -4.$

8. $3x + 2y = 13,$
$3y - 2z = 7,$
$z + w = 2,$
$w - 3y = -17.$

9. $x - y + z = 2,$
$y - z + w = 3,$
$z - w + x = 0,$
$w - x + y = 5.$

10. $2v + 3x + 2y - z + w = 8,$
$3v - 3y + 2w = -8,$
$v - w + x = 2,$
$w + x - y = 0,$
$3w + 2v - 7x = -10.$

CHAPTER XX

PARTIAL FRACTIONS

193. Introduction. For various purposes it is convenient to express a rational algebraic expression $\frac{f(x)}{F(x)}$, section 11, as the sum of several fractions called **partial fractions**, which have the several factors of $F(x)$ as denominators and which have constants for numerators. If we write

$$F(x) = (x - a)(x - b) \cdots (x - n),$$

we seek a means of determining constants $A, B, \cdots, N$ such that for every value of x

$$\frac{f(x)}{F(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \cdots + \frac{N}{x-n}. \tag{1}$$

If the degree of $f(x)$ is equal to or greater than that of $F(x)$, we can write

$$\frac{f(x)}{F(x)} = Q(x) + \frac{f_1(x)}{F(x)}, \tag{2}$$

where $Q(x)$ is the quotient and $f_1(x)$ the remainder from dividing $f(x)$ by $F(x)$, and where the degree of $f_1(x)$ is less than that of $F(x)$. In what follows we shall assume that the degree of $f(x)$ is less than that of $F(x)$. In problems where this is not the case one should carry out the long division indicated by (2) and apply the principles developed in this chapter to the expression corresponding to the last term in (2).

194. Development when $F(x) = 0$ has no multiple roots. Let us consider the particular case

$$\frac{f(x)}{F(x)} = \frac{x+1}{(x-1)(x-2)(x-3)}.$$

We indicate the development required in form (1) of the last section,

$$\frac{x+1}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}, \quad (1)$$

where A, B, and C represent constants which we are to determine if possible. The question arises immediately, Are we at liberty to make this assumption? Are we not assuming the essence of what we wish to prove, that is, the form of the expansion? To this we may answer, We have written the expansion in form (1) tentatively. We have not proved it and are not certain of its validity. If, however, we are able to find numerical values of A, B, and C which satisfy (1), we can then actually write down the development of the fraction in the form of an identity.

If, on the other hand, we can show that no such numbers A, B, C satisfying (1) exist, then the development is not possible.

Clear (1) of fractions:

$$\begin{aligned} x+1 &= A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2) \\ &= (A+B+C)x^2-(5\,A+4\,B+3\,C)x+6\,A+3\,B+2\,C. \end{aligned}$$

Since we seek values of A, B, and C for which (1) is identically true for all values of x, equate coefficients of like powers of x in the last equation (Corollary II, p. 192). We obtain

$$A+B+C=0, \quad (2)$$

$$-5\,A-4\,B-3\,C=1, \quad (3)$$

and

$$6\,A+3\,B+2\,C=1. \quad (4)$$

Add (4) to (3) and we obtain

$$A-B-C=2 \quad (5)$$

$$\underline{A+B+C=0} \quad (2)$$

Adding, we obtain

$$2\,A=2$$

$$A=1.$$

From (3) and (4) we obtain

$$B=-3, \text{ and } C=2.$$

Thus $$\frac{x+1}{(x-1)(x-2)(x-3)}=\frac{1}{x-1}-\frac{3}{x-2}+\frac{2}{x-3}.$$

As a check we might clear of fractions and simplify. If equations (2), (3), and (4) had been incompatible, we should have concluded that we could not develop the fraction in form (1).

We assume now for the general case that

$$F(x)=(x-a)(x-b)\cdots(x-n),$$

and that the n roots $a, b, \cdots, n$ are all distinct from each other.

Let us consider the expression

$$\frac{f(x)}{F(x)}=\frac{A}{x-a}+\frac{B}{x-b}+\cdots+\frac{N}{x-n}, \tag{6}$$

where A, B, $\cdots$, N are constants. Let us assume for the moment the possibility of expressing $\frac{f(x)}{F(x)}$ in terms of these partial fractions. We shall now attempt to determine actual values A, B, $\cdots$, N which satisfy such an identity. If we multiply both sides of the identity by

$$F(x)=(x-a)(x-b)\cdots(x-n),$$

we obtain

$$f(x)=A(x-b)\cdots(x-n)+B(x-a)\cdots(x-n)+\cdots$$
$$+N(x-a)(x-b)\cdots.$$

$f(x)$ is of degree not greater than $n-1$, and consequently when written in the form of (1), p. 184, has not more than n terms. If we multiply out the right-hand member and collect powers of x, we have an expression in x of degree $n-1$. By Corollary II, p. 192, this equation will be an identity if we can determine values of A, B, $\cdots$, N which make the coefficients of x on both sides of the equation equal to each other. Hence we equate coefficients of like powers of x and obtain n equations linear in A, B, $\cdots$, N which we can treat as variables. These equations have in general one and only one solution, which we can easily determine. The values of

$A, B, \cdots, N$ obtained by solving these equations we can substitute for the numerators of the partial fractions in (6). After making this substitution we can actually clear of fractions the right-hand member of (6) and check our work by showing its identity with the left-hand member.

There is no general criterion that we have applied to (6) to determine whether the n linear equations obtained by equating coefficients of x have any solution or not. Hence in this general discussion it should be distinctly understood that assumption (6) holds when and only when these equations are solvable. In any particular case we can find out immediately whether the equations are solvable by attempting to solve them. If the numbers $A, B, \cdots, N$ do not exist, the fact will appear by our inability to solve the linear equations. As a matter of fact, one and only one solution always exists under the assumption of this section.

If in (6) we assume that several of the symbols $A, B, \cdots, N$ stand for expressions linear in x, as, for instance, $ax + b$, we should then have a larger number of variables to determine than there are equations. Under these circumstances there is an infinite number of solutions of the equations. Thus if we should seek to express $\frac{f(x)}{F(x)}$ as the sum of partial fractions where the numerators are not constants but functions of x, we could get any number of such developments.

We have the following

Rule. *Factor $F(x)$ into linear factors, as*

$$(x-a)(x-b)\cdots(x-n).$$

Write the expression

$$\frac{f(x)}{F(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \cdots + \frac{N}{x-n}.$$

Multiply both sides of the expression by $F(x)$, equate coefficients of like powers of x, and solve the resulting linear equations for $A, B, \cdots, N$.

Replace $A, B, \cdots, N$ by these values and check by substituting for x some number distinct from $a, b, \cdots, n$.

EXERCISES

Separate into partial fractions:

1. $\dfrac{x^2-2}{(x-1)(x-2)x}$.

Solution. Assume $$\frac{x^2-2}{(x-1)(x-2)x}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x}. \qquad (1)$$

Multiplying by $(x-1)(x-2)x$,

$$\begin{aligned} x^2-2 &= A(x-2)x+B(x-1)x+C(x-1)(x-2)\\ &=(A+B+C)x^2-(2A+B+3C)x+2C. \end{aligned}$$

Equating coefficients of like powers of x,

$$A+B+C=1,$$

$$2A+B+3C=0,$$

and $$2C=-2.$$

Hence $$C=-1.$$

Solving $$A+B=2$$
and $$2A+B=3$$
we obtain $$A=1$$

Hence $$B=1.$$

Thus $$\frac{x^2-2}{(x-1)(x-2)x}=\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x}.$$

Check. Let $x=-1$.

Substituting in (1), $$\frac{-1}{-6}=\frac{1}{-2}+\frac{1}{-3}-\frac{1}{-1},$$

or $$\frac{1}{6}=-\frac{1}{2}-\frac{1}{3}+1=-\frac{5}{6}+1=\frac{1}{6}.$$

2. $\dfrac{x-1}{x^2+3x+2}$.

3. $\dfrac{x+7}{2x^2-5x-3}$.

4. $\dfrac{1}{3x^2-2x-8}$.

5. $\dfrac{5x}{6x^2-5x-1}$.

6. $\dfrac{4x^2}{(x^2-4)(x-1)}$.

7. $\dfrac{2x^2-1}{(x^2+3x+2)(x-1)}$.

8. $\dfrac{x^2-3x+1}{(x-1)(x-2)(x-3)}$.

9. $\dfrac{x^2+4}{(x-2)(x+2)(x-1)}$.

195. Development when $F(x) = 0$ has imaginary roots. In the preceding section no mention has been made of any distinction between real and imaginary values of $a, b, \cdots, n$. In fact the method given is valid whether they are real or imaginary. It is, however, desirable to obtain a development in which only real numbers appear.

Let us assume the development

$$\frac{f(x)}{F(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \cdots + \frac{M}{x-m} + \frac{N}{x-n}, \tag{1}$$

where let us suppose that m and n are *the only* pair of conjugate imaginary roots of $F(x)$, M and N being conjugate complex numbers.

Let $$m = p + ir, \quad n = p - ir.$$

Then, adding the corresponding terms of (1), we obtain

$$\frac{M}{x-p-ir} + \frac{N}{x-p+ir}$$
$$= \frac{x(M+N) - p(M+N)}{(x-p)^2 + r^2} + i\,\frac{r(M-N)}{(x-p)^2 + r^2}. \tag{2}$$

Since m and n are the only imaginary roots of $F(x) = 0$, the last term of (2) is real, as is also the entire right-hand member (section 142). Hence, letting the numerator

$$x(M+N) - p(M+N) + ir(M-N) = Px + R,$$

we have the development

$$\frac{f(x)}{F(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \cdots + \frac{Px+R}{(x-p)^2 + r^2}. \tag{3}$$

By complete induction we can establish this form of numerator where there is any number of pairs of imaginary roots of $F(x) = 0$. We have proved the form (3) where there is one pair of imaginary roots. Assuming the form where there are k pairs, we can prove it similarly where there are $k+1$ pairs. Hence we have the

Theorem. If $F(x)$ is factorable into distinct linear and quadratic factors, but the quadratic factors are not further reducible *into real* factors, then* $\frac{f(x)}{F(x)}$ *is separable into partial fractions of the form*

$$\frac{A}{x-a}+\frac{B}{x-b}+\cdots+\frac{Px+R}{x^2+mx+n},$$

where x^2+mx+n *is an irreducible quadratic factor of* $F(x)$.

This theorem is of course true only under the condition that the linear equations obtained in the process of determining the constants are solvable. It turns out, however, that in this case, as in section 194, the linear equations obtained always have one and only one solution provided that the roots are all distinct.

EXERCISES

Separate into partial fractions:

1. $\frac{x^2+1}{(x-1)(x^2+x+1)}$.

Solution. Assume $\frac{x^2+1}{(x-1)(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$.

Multiplying by $(x-1)(x^2+x+1)$,

$$x^2+1=A(x^2+x+1)+(Bx+C)(x-1).$$

Collecting like powers of x,

$$x^2+1=(A+B)x^2+(A-B+C)x+A-C.$$

Equating coefficients of x, $A+B=1$, (1)

$$A-B+C=0, \quad (2)$$

$$A-C=1. \quad (3)$$

Adding (2) and (3), and solving with (1),

$$\begin{array}{r} 2A-B=1 \\ \underline{A+B=1} \\ 3A=2 \end{array}$$

or $A=\frac{2}{3}$.

Substituting in (1), $\frac{2}{3}+B=1$,

or $B=\frac{1}{3}$.

---

*A *real factor* is one whose coefficients are all real numbers.

Substituting in (3), $\frac{2}{3} - C = 1,$

or $C = -\frac{1}{3}.$

Thus $$\frac{x^2+1}{(x-1)(x^2+x+1)} = \frac{2}{3(x-1)} + \frac{x-1}{3x^2+3x+3}.$$

Check. Let $x = -1.$

Substituting, $$\frac{2}{-2\cdot 1} = \frac{2}{3\cdot -2} + \frac{-2}{3-3+3}.$$

Reducing, $$-1 = -\frac{2}{6} - \frac{2}{3} = -1.$$

2. $\dfrac{x^2+x+1}{x^3+4x}.$

3. $\dfrac{x^2+1}{x^4+x^2+1}.$

4. $\dfrac{x^2+4}{x^3-2x^2+3x-2}.$

5. $\dfrac{5x^3-1}{x^4+6x^2+8}.$

6. $\dfrac{x}{(x+3)(2x^2-x-4)}.$

7. $\dfrac{1}{x^3+3x^2-2x-16}.$

HINT. Factor by synthetic division.

8. $\dfrac{x^2+1}{(x-1)(3x^2+x+6)}.$

196. Development when $F(x) = (x-a)^n$. In this case the method given in the previous sections fails, as the equations for determining the values of the numerators are incompatible. If we let

$$\frac{f(x)}{F(x)} = \frac{a_0x^{n-1} + a_1x^{n-2} + \cdots + a_{n-1}}{(x-a)^n}, \qquad (1)$$

we can separate into partial fractions as follows.

Let $x - a = y$, that is, $x = y + a$, and substitute in (1). We obtain, after collecting powers of y,

$$\frac{A_0y^{n-1} + A_1y^{n-2} + \cdots + A_{n-1}}{y^n} = \frac{A_0}{y} + \frac{A_1}{y^2} + \cdots + \frac{A_{n-1}}{y^n},$$

where the A's are constants. Replacing y by $x - a$, we have the following development:

$$\frac{f(x)}{(x-a)^n} = \frac{A_0}{x-a} + \frac{A_1}{(x-a)^2} + \frac{A_2}{(x-a)^3} + \cdots + \frac{A_{n-1}}{(x-a)^n}.$$

EXERCISES

Separate into partial fractions:

1. $\dfrac{x^2+2x-1}{(x-3)^3}$.

Solution. Assume $$\frac{x^2+2x-1}{(x-3)^3}=\frac{A}{x-3}+\frac{B}{(x-3)^2}+\frac{C}{(x-3)^3}. \qquad (1)$$

Multiplying by $(x-3)^3$,

$$\begin{aligned} x^2+2x-1 &= A(x^2-6x+9)+B(x-3)+C \\ &= Ax^2+(B-6A)x+9A-3B+C. \end{aligned}$$

Equating coefficients of x,

$$A=1,$$

$$B-6A=2,$$

and $$9A-3B+C=-1.$$

Solving, $$B=8 \text{ and } C=14.$$

Hence $$\frac{x^2+2x-1}{(x-3)^3}=\frac{1}{x-3}+\frac{8}{(x-3)^2}+\frac{14}{(x-3)^3}.$$

Check. Let $x=1$.

Substituting in (1), $$\frac{2}{-8}=\frac{1}{-2}+\frac{8}{4}-\frac{14}{8}=-\frac{1}{2}+2-\frac{14}{8},$$

or $$-\frac{1}{4}=-\frac{1}{4}.$$

2. $\dfrac{x^2-4}{(x-2)^3}$.

3. $\dfrac{x}{(2x+1)^2}$.

4. $\dfrac{x}{(x-4)^3}$.

5. $\dfrac{x^2+x+1}{(2x-1)^4}$.

6. $\dfrac{x-a}{(ax+b)^2}$.

7. $\dfrac{2x^2+3x+1}{(3x-2)^3}$.

197. General case. When $F(x)=0$ has real, complex, and multiple roots, we may use all the previous methods simultaneously. Hence for this case we assume the expansion

$$\frac{f(x)}{(x-a)\cdots(lx^2+mx+n)\cdots(x-t)^k}$$
$$=\frac{A}{x-a}+\cdots+\frac{Px+R}{lx^2+mx+n}+\cdots+\frac{T}{x-t}+\cdots+\frac{V}{(x-t)^k}.$$

EXERCISES

Separate into partial fractions:

1. $\dfrac{x^4 + 2x^2 + 18x - 18}{(x-1)(x^2+x+1)(x-2)^2}$.

Solution.

$$\frac{x^4 + 2x^2 + 18x - 18}{(x-1)(x^2+x+1)(x-2)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} + \frac{D}{x-2} + \frac{E}{(x-2)^2}.$$

Clearing of fractions,

$$\begin{aligned} x^4 + 2x^2 + 18x - 18 &= A(x^2+x+1)(x-2)^2 + (Bx+C)(x-2)^2(x-1) \\ &\quad + D(x-1)(x-2)(x^2+x+1) \\ &\quad + E(x-1)(x^2+x+1) \\ &= (A+B+D)x^4 + (-3A-5B+C-2D+E)x^3 \\ &\quad + (A+8B-5C)x^2 + (-4B+8C-D)x \\ &\quad + (4A-4C+2D-E). \end{aligned}$$

Equating coefficients of like powers of x,

$$A + B + D = 1, \qquad (1)$$

$$-3A - 5B + C - 2D + E = 0, \qquad (2)$$

$$A + 8B - 5C = 2, \qquad (3)$$

$$-4B + 8C - D = 18, \qquad (4)$$

$$4A - 4C + 2D - E = -18. \qquad (5)$$

Adding (2) and (5), (1) and (4), we have, together with (3),

$$A - 5B - 3C = -18, \qquad (6)$$

$$A - 3B + 8C = 19, \qquad (7)$$

and

$$A + 8B - 5C = 2. \qquad (3)$$

Adding all three equations, we find

$$3A = 3, \text{ or } A = 1.$$

Substituting in (3) and (7), and solving, we find $C = 3$ and $B = 2$. Substituting in (1), we find $D = -2$. Similarly, from (5), $E = 6$.

Thus

$$\frac{x^4 + 2x^2 + 18x - 18}{(x-1)(x^2+x+1)(x-2)^2} = \frac{1}{x-1} + \frac{2x+3}{x^2+x+1} - \frac{2}{x-2} + \frac{6}{(x-2)^2}.$$

Check. Let $x = -1$.

Substituting, $$\frac{-33}{-2 \cdot 1 \cdot 9} = \frac{1}{-2} + \frac{1}{1} - \frac{2}{-3} + \frac{6}{9},$$

or $$\tfrac{11}{6} = 2\tfrac{1}{3} - \tfrac{1}{2} = \tfrac{11}{6}.$$

2. $\dfrac{x^3+1}{x(x-1)^3}$.

3. $\dfrac{4\,x^2}{1-x^4}$.

4. $\dfrac{x^3+2\,x^2+1}{x(x-1)^3}$.

5. $\dfrac{5\,x+12}{x(x^2+4)}$.

6. $\dfrac{2\,x^3+x+3}{(x^2+1)^2}$.

7. $\dfrac{43\,x-11}{30(x^2-1)}$.

8. $\dfrac{x-1}{(x+1)^2(x+2)}$.

9. $\dfrac{x^3-x-1}{x^4-16}$.

10. $\dfrac{11\,x+7}{(x^2-1)(x+2)}$.

11. $\dfrac{x^2+6\,x-8}{x^3-4\,x}$.

12. $\dfrac{x^2-6\,x+1}{(x-8)(x-9)}$.

13. $\dfrac{2}{(x^2+x+3)(2\,x+1)}$.

14. $\dfrac{2\,x^2-3\,x-3}{(x-1)(x^2-2\,x+5)}$.

15. $\dfrac{x^3+2\,x^2-3\,x+1}{(x+3)(x^2-4\,x+5)}$.

16. $\dfrac{30\,x-17}{(2\,x-3)(6\,x^2-5\,x+1)}$.

17. $\dfrac{13-5\,x}{(x^2-3\,x+2)(x-3)}$.

18. $\dfrac{5+2\,x}{(x^2+2\,x+1)\,(x-1)}$.

CHAPTER XXI

CONTINUED FRACTIONS

198. Definitions. A fraction in the form

$$a+\cfrac{b}{c+\cfrac{d}{e+\cfrac{f}{g+\cdots,}}}$$

where $a, b, \cdots, g, \cdots$ are real numbers, is called a **continued fraction.** We shall consider only those continued fractions in which the numerators b, d, f, etc. are equal to unity and in which the letters represent integers, as, for example,

$$a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\cdots,}}} \qquad \text{written } a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\cdots.$$

When the number of quotients $a_2, a_3, a_4, \cdots$ is finite the fraction is said to be **terminating.** When the fraction is not terminating it is **infinite.** We shall see that the character of the numbers represented by terminating fractions differs widely from that of the numbers represented by infinite continued fractions. We shall find, in fact, that any root of a linear equation in one variable, that is, any rational number, may be represented by a terminating continued fraction, and conversely; furthermore, that any real irrational root of a quadratic equation may be represented by the simplest type of infinite continued fractions, and conversely.

199. Terminating continued fractions. If we have a terminating continued fraction, where $a_1, a_2, \cdots$ are integers, it is evident that by reducing it to its simplest form we obtain a rational number. The converse is also true, as we can prove in the following

Theorem. *Any rational number may be expressed as a terminating fraction.*

Let $\frac{a}{b}$ represent a rational number. Divide a by b, and let a_1 be the quotient and c (less than b) the remainder. Then

$$\frac{a}{b} = a_1 + \frac{c}{b} = a_1 + \cfrac{1}{\cfrac{b}{c}}.$$

Divide b by c, letting a_2 be the quotient and d (which must be less than c) the remainder. Then

$$\frac{a}{b} = a_1 + \cfrac{1}{a_2 + \cfrac{d}{c}}.$$

Continuing this process, the maximum limit of the remainders in the successive divisions becomes smaller as we go on, until finally the remainder is zero. Hence the fraction is terminating. It is noted that the successive *quotients* are the denominators in the continued fraction.

EXERCISES

Convert the following into continued fractions:

1. $\frac{77}{247}$.

Solution.
$$\begin{array}{l}
\underline{247|}\,77\,\underline{|0}\\
\quad\underline{77|}\,247\,\underline{|3}\\
\qquad\quad\ \underline{231}\\
\qquad\quad\underline{16|}\,77\,\underline{|4}\\
\qquad\qquad\quad\ \underline{64}\\
\qquad\qquad\quad\underline{13|}\,16\,\underline{|1}\\
\qquad\qquad\qquad\quad\underline{13}\\
\qquad\qquad\qquad\quad\underline{3|}\,13\,\underline{|4}\\
\qquad\qquad\qquad\qquad\ \ \underline{12}\\
\qquad\qquad\qquad\qquad\ \ \underline{1|}\,3\,\underline{|3}\\
\qquad\qquad\qquad\qquad\qquad\underline{3}\\
\qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

The continued fraction is

$$\frac{77}{247}=\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{3}.$$

2. $\frac{13}{97}$.
3. $\frac{61}{437}$.
4. $\frac{204}{457}$.
5. $\frac{115}{151}$.
6. $\frac{408}{985}$.
7. $\frac{613}{2363}$.
8. $\frac{1500}{2149}$.
9. $\frac{505}{192}$.
10. $\frac{9697}{1557}$.

Express the following continued fractions as rational fractions.

11. $\frac{1}{2+}\,\frac{1}{3}$.
12. $\frac{1}{4+}\,\frac{1}{7}$.
13. $\frac{1}{a+}\,\frac{1}{b}$.
14. $\frac{1}{x+}\,\frac{1}{x}$.
15. $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3}$.
16. $\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{5}$.
17. $\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2}$.
18. $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3}$.

200. Convergents. The value obtained by taking only the first $n-1$ quotients in a continued fraction is called the nth **convergent** of the fraction.

Thus in the fraction $1+\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{6}$

1 is the first convergent,

$1+\frac{1}{2}=\frac{3}{2}$ is the second convergent,

$1+\frac{1}{2+}\,\frac{1}{3}=1+\frac{3}{7}=\frac{10}{7}$ is the third convergent, etc.

When there is no whole number preceding the fractional part of the continued fraction the first convergent is zero. Thus in

$$\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{5}$$

$\frac{1}{2}$ is called the second convergent.

In the continued fraction

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\,\frac{1}{a_5+}\cdots$$

let $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \cdots$ represent the successive convergents expressed as rational fractions.

Then for the first convergent we have

$$a_1 = \frac{p_1}{q_1}, \text{ or } p_1 = a_1,\ q_1 = 1.$$

For the second convergent we have

$$a_1 + \frac{1}{a_2} = \frac{a_2a_1 + 1}{a_2} = \frac{p_2}{q_2}, \text{ or } p_2 = a_2a_1 + 1 = a_2p_1 + 1,$$
$$q_2 = a_2 = a_2q_1.$$

For the third convergent we have

$$a_1 + \frac{1}{a_2 +} \frac{1}{a_3} = a_1 + \frac{a_3}{a_3a_2 + 1} = \frac{a_3(a_2a_1 + 1) + a_1}{a_3a_2 + 1} = \frac{p_3}{q_3},$$

or

$$p_3 = a_3(a_2a_1 + 1) + a_1 = a_3p_2 + p_1,$$
$$q_3 = a_3a_2 + 1 = a_3q_2 + q_1.$$

This indicates that the form of the nth convergent is

$$\frac{p_n}{q_n} = \frac{a_np_{n-1} + p_{n-2}}{a_nq_{n-1} + q_{n-2}}. \tag{1}$$

This is in fact the case, as we proceed to show by complete induction.

We have already established form (1) for $n = 2$ and $n = 3$. We assume it for $n = m$, and will show that its validity for $n = m + 1$ follows. The $(m + 1)$th convergent differs from the mth only in the fact that $a_m + \dfrac{1}{a_{m+1}}$ appears in the continued fraction in place of a_m. In (1) replace n by m, and a_n by $a_m + \dfrac{1}{a_{m+1}}$, and we have

$$\frac{p_{m+1}}{q_{m+1}} = \frac{\left(a_m + \dfrac{1}{a_{m+1}}\right)p_{m-1} + p_{m-2}}{\left(a_m + \dfrac{1}{a_{m+1}}\right)q_{m-1} + q_{m-2}}$$
$$= \frac{(a_{m+1}a_m + 1)p_{m-1} + a_{m+1}p_{m-2}}{(a_{m+1}a_m + 1)q_{m-1} + a_{m+1}q_{m-2}}$$
$$= \frac{a_{m+1}(a_mp_{m-1} + p_{m-2}) + p_{m-1}}{a_{m+1}(a_mq_{m-1} + q_{m-2}) + q_{m-1}}$$
$$= \frac{a_{m+1}p_m + p_{m-1}}{a_{m+1}q_m + q_{m-1}}, \text{ which is form (1).}$$

EXERCISES

Express the following as continued fractions, and find the convergents:

1. $\frac{30}{41}$.

Solution. By the method already explained, we find that

$$\frac{30}{41}=\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2}.$$

Here $a_1=0$, $a_2=1$, $a_3=2$, $a_4=1$, $a_5=2$, $a_6=1$, $a_7=2$.

The first convergent is evidently 0, the second is 1, and the third is $\dfrac{1}{1+\frac{1}{2}}=\dfrac{2}{3}$.

The fourth convergent is $\dfrac{p_4}{q_4}=\dfrac{a_4p_3+p_2}{a_4q_3+q_2}=\dfrac{1\cdot 2+1}{1\cdot 3+1}=\dfrac{3}{4}$.

The fifth convergent is $\dfrac{p_5}{q_5}=\dfrac{a_5p_4+p_3}{a_5q_4+q_3}=\dfrac{2\cdot 3+2}{2\cdot 4+3}=\dfrac{8}{11}$.

The sixth convergent is $\dfrac{p_6}{q_6}=\dfrac{a_6p_5+p_4}{a_6q_5+q_4}=\dfrac{1\cdot 8+3}{1\cdot 11+4}=\dfrac{11}{15}$.

The seventh convergent is $\dfrac{p_7}{q_7}=\dfrac{a_7p_6+p_5}{a_7q_6+q_5}=\dfrac{2\cdot 11+8}{2\cdot 15+11}=\dfrac{30}{41}$.

2. $\frac{65}{352}$.
3. $\frac{55}{89}$.
4. $\frac{36}{121}$.
5. $\frac{57}{161}$.
6. $\frac{100}{137}$.
7. $\frac{75}{194}$.
8. $\frac{241}{1760}$.
9. $\frac{56}{457}$.
10. $\frac{900}{3361}$.
11. $\frac{212}{313}$.

Find the value of the following by finding the successive convergents:

12. $\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2}$.
13. $\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\frac{1}{3}$.
14. $\frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3+}\frac{1}{2}$.
15. $\frac{1}{3+}\frac{1}{3+}\frac{1}{3+}\frac{1}{3+}\frac{1}{3+}\frac{1}{3}$.
16. $\frac{1}{5+}\frac{1}{3+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3+}\frac{1}{5}$.
17. $\frac{1}{1+}\frac{1}{3+}\frac{1}{5+}\frac{1}{5+}\frac{1}{3+}\frac{1}{1}$.
18. $\frac{1}{(x-1)+}\frac{1}{x+}\frac{1}{(x+1)}$.
19. $\frac{1}{x+}\frac{1}{x+}\frac{1}{x}$.

201. Recurring continued fractions. We have seen that every terminating continued fraction represents a rational number, and conversely. We now discuss the character of the numbers represented by the simplest infinite continued fractions. A **recurring** continued fraction is one in which from a certain point on, a group of denominators is repeated in the same order.

Thus $$\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\cdots,$$

and $$\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\cdots$$

are recurring continued fractions if the denominators are assumed to repeat indefinitely as indicated.

That a repeating continued fraction actually represents a number we shall establish in section 203. Unless this fact is proven, one runs the risk of dealing with symbols which have no meaning. If for certain continued fractions the successive convergents increase without limit, or take on erratic values that approach no limit, it is important to discover the fact. All the fractions that we discuss actually represent numbers, as we shall see.

We shall consider only continued fractions in which every denominator has a positive sign.

Theorem. Every recurring continued fraction is the root of a quadratic equation.

Let, for instance,

$$x=\frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\cdots.$$

Evidently the part of the fraction after the first denominator c may be represented by x, and we have thus virtually the terminating fraction

$$x=\frac{1}{a+}\frac{1}{b+}\frac{1}{c+x}.$$

The second convergent is $\frac{1}{a}$.

The third convergent is

$$\frac{b}{ab+1}=\frac{p_3}{q_3}.$$

The fourth convergent, or x, gives us

$$x = \frac{p_4}{q_4} = \frac{a_4p_3 + p_2}{a_4q_3 + q_2} = \frac{(c + x)b + 1}{(c + x)(ab + 1) + a}.$$

Simplifying, we get

$$(ab + 1)x^2 + [c(ab + 1) + a - b]x - bc - 1 = 0,$$

which is a quadratic equation whose root is x, the value of the continued fraction.

Since this equation has a negative number for its constant term, it has one positive and one negative root. The continued fraction must represent the positive root, since we assume that the letters a, b, c represent positive integers. The quadratic equation whose root is a recurring continued fraction with positive denominators will always have one positive and one negative root. The equation will be quadratic, however, whatever the signs of the denominators may be.

The proof may be extended to the case where there are any number of recurring denominators or any number of denominators before the recurrence sets in. Since every real irrational root of a quadratic equation is a surd, our result is equivalent to the statement that every recurring continued fraction may be expressed as a surd.

EXERCISES

Of what quadratic equations are the following roots? Express the continued fraction as a surd.

1. $\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\cdots.$

Solution. Let $x = \frac{1}{2+}\,\frac{1}{3+x}.$

Then $$x = \frac{1}{2+}\,\frac{1}{3+x} = \frac{3+x}{6+2x+1},$$

or $$2x^2 + 6x - 3 = 0.$$

Solving this equation, we get

$$x_1 = \frac{-3 + \sqrt{15}}{2} \quad \text{or} \quad x_2 = \frac{-3 - \sqrt{15}}{2}.$$

Since x_2 is negative, x_1 must be the surd that is represented by the continued fraction.

Thus
$$\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\cdots=\frac{-3+\sqrt{15}}{2}.$$

2. $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots.$

3. $\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\cdots.$

4. $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\cdots.$

5. $\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\cdots.$

6. $2+\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\cdots.$

HINT. If $x=2+\frac{1}{2+}\,\frac{1}{1+}\cdots,$

then $x-2=\frac{1}{2+}\,\frac{1}{1+}\cdots$

and $x=2+\frac{1}{2+}\,\frac{1}{1+(x-2)}.$

7. $3+\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{5+}\cdots.$

8. $1+\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{3+}\,\frac{1}{4+}\cdots.$

9. $1+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\cdots.$

10. $\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots.$

202. Expression of a surd as a recurring continued fraction. This is the converse of the problem discussed in the last section, and shows that recurring continued fractions and quadratic equations are related in the same intimate way that terminating fractions and rational numbers (that is, the roots of linear equations) are connected. We seek to express an irrational number, as, for instance, $\sqrt{2}$, as a continued fraction. This we may do as follows.

Since 1 is the largest integer in $\sqrt{2}$, we may write

$$\sqrt{2}=1+(\sqrt{2}-1)=1+\frac{\sqrt{2}-1}{1}.$$

Rationalizing the numerator, we have

$$\sqrt{2}=1+\frac{1}{\sqrt{2}+1}.$$

Since 2 is the largest integer in $\sqrt{2}+1$, we may write this expression as follows

$$\sqrt{2} = 1 + \frac{1}{2 + (\sqrt{2}-1)} = 1 + \frac{1}{2 + \dfrac{\sqrt{2}-1}{1}}.$$

Rationalizing the numerator $\sqrt{2}-1$, we obtain the following expression:

$$\sqrt{2} = 1 + \frac{1}{2+}\,\frac{1}{\sqrt{2}+1} = 1 + \frac{1}{2+}\,\frac{1}{2+}\,(\sqrt{2}-1).$$

By continuing this process we continually get the denominator 2. Thus

$$\sqrt{2} = 1 + \frac{1}{2+}\,\frac{1}{2+}\cdots.$$

This process consists of the successive application of two operations, and affords the following

Rule. *Express the surd as the sum of two numbers the first of which is the largest integer that it contains.*

Rationalize the numerator of the fraction whose numerator is the second of these numbers. Repeat these operations until a recurrence of denominators is observed.

This process may be applied to any surd, and a continued fraction which is recurring will always be obtained. We shall content ourselves with a statement of this fact without proof.

If this surd is of the form $a-\sqrt{b}$, a continued fraction may be derived for $+\sqrt{b}$ and its sign changed. Since the real roots of any quadratic equation $x^2 + 2\,a_1x + a_2 = 0$ are surds of the form $a \pm \sqrt{b}$, where a and b are integers, it appears that the roots of any such equation may be expressed as recurring continued fractions. It can be shown that the real roots of the general quadratic equation $a_0x^2 + a_1x + a_2 = 0$ may also be so expressed.

EXERCISES

Express the following surds as recurrent continued fractions.

1. $2+\sqrt{3}$.

Solution.

$$2+\sqrt{3}=3+(\sqrt{3}-1)=3+\frac{\sqrt{3}-1}{1} \qquad (1)$$

$$=3+\frac{3-1}{\sqrt{3}+1}=3+\frac{2}{\sqrt{3}+1}$$

$$=3+\frac{1}{\dfrac{\sqrt{3}+1}{2}}=3+\frac{1}{1+\dfrac{\sqrt{3}+1}{2}-1}=3+\frac{1}{1+\dfrac{\sqrt{3}-1}{2}}$$

$$=3+\frac{1}{1+}\,\frac{3-1}{2(\sqrt{3}+1)}=3+\frac{1}{1+}\,\frac{1}{\dfrac{\sqrt{3}+1}{1}}=3+\frac{1}{1+}\,\frac{1}{2+}(\sqrt{3}-1).$$

But since $\sqrt{3}-1$ is the same number that we have in (1), this fraction repeats from this point on, and we have

$$2+\sqrt{3}=3+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots.$$

2. $\sqrt{5}$.
3. $\sqrt{17}$.
4. $\sqrt{65}$.
5. $\sqrt{47}$.
6. $\sqrt{14}$.
7. $\sqrt{23}$.
8. $\sqrt{34}$.
9. $\sqrt{19}$.
10. $\sqrt{62}$.
11. $\sqrt{79}$.
12. $\sqrt{98}$.
13. $\sqrt{88}$.
14. $\sqrt{22}$.
15. $\sqrt{45}$.
16. $\sqrt{59}$.
17. $\sqrt{101}$.
18. $7+\sqrt{11}$.
19. $8-\sqrt{3}$.
20. $3-\sqrt{23}$.

Express as a continued fraction the roots of the following equations:

21. $x^2-7x-3=0$.
22. $x^2+2x-6=0$.
23. $x^2+3x-8=0$.
24. $x^2-4x-4=0$.

203. Properties of convergents. The law of formation of convergents given in section 200 is valid whether the continued fraction is terminating or infinite. We should expect that in the case of an infinite fraction the successive convergents would give us an increasingly close approximation to the value of the fraction. This is indeed the fact, as we shall see.

Theorem. *The difference between the nth and $(n+1)$st convergents is* $\dfrac{(-1)^{n+1}}{q_nq_{n+1}}$.

We prove this theorem by mathematical induction.

Let the continued fraction be

$$a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\cdots.$$

Then the first and second convergents are, respectively,

$$\frac{p_1}{q_1} = a_1 \quad\text{and}\quad \frac{p_2}{q_2} = \frac{a_2a_1+1}{a_2}.$$

Then $$\frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{a_2a_1+1}{a_2} - a_1 = \frac{1}{a_2}.$$

Since $$q_1 = 1,\ q_2 = a_2,$$

we have $$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^{n+1}}{q_nq_{n+1}} \text{ for } n = 1.$$

We assume that the theorem holds for $n = m$; that is,

$$\frac{p_{m+1}}{q_{m+1}} - \frac{p_m}{q_m} = \frac{p_mq_{m+1} - q_mp_{m+1}}{q_mq_{m+1}} = \frac{(-1)^{m+1}}{q_mq_{m+1}}. \tag{1}$$

We must prove that it holds for $n = m+1$.

Now since

$$\frac{p_{m+2}}{q_{m+2}} - \frac{p_{m+1}}{q_{m+1}} = \frac{p_{m+1}q_{m+2} - p_{m+2}q_{m+1}}{q_{m+1}q_{m+2}},$$

our theorem reduces to proving that the numerator

$$p_{m+1}q_{m+2} - p_{m+2}q_{m+1} = (-1)^{m+2}. \tag{2}$$

In the left-hand member of (2), set

$$a_{m+2}q_{m+1} + q_m = q_{m+2},$$

and $$a_{m+2}p_{m+1} + p_m = p_{m+2}.$$

Then

$$\begin{aligned} &p_{m+1}(a_{m+2}q_{m+1} + q_m) - (a_{m+2}p_{m+1} + p_m)q_{m+1} \\ &\quad = p_{m+1}q_m - p_mq_{m+1} = -(p_mq_{m+1} - p_{m+1}q_m) \\ &\quad = -(-1)^{m+1} = (-1)^{m+2}. \qquad \text{By (1)} \end{aligned}$$

Corollary I. The difference between the successive convergents of a continued fraction with positive denominators approaches zero as a limit.

Since $q_n = a_n q_{n-1} + q_{n-2}$, evidently q_n increases without limit when n is increased, since to obtain q_n we add together positive numbers neither one of which can vanish.

Thus we can find a value of n large enough so that $\frac{1}{q_n}$, and hence $\frac{1}{q_n q_{n+1}}$, will be smaller than any assigned number, which is another way of stating that as n increases $\frac{1}{q_n q_{n+1}}$ approaches zero as a limit.

Corollary II. The even convergents decrease, while the odd convergents increase, as n increases.

We must show that

$$\frac{p_{m+2}}{q_{m+2}} - \frac{p_m}{q_m}$$

is negative or positive according as m is even or odd. Adding and subtracting $\frac{p_{m+1}}{q_{m+1}}$, we have

$$\frac{p_{m+2}}{q_{m+2}} - \frac{p_m}{q_m} = \left(\frac{p_{m+2}}{q_{m+2}} - \frac{p_{m+1}}{q_{m+1}}\right) + \left(\frac{p_{m+1}}{q_{m+1}} - \frac{p_m}{q_m}\right)$$
$$= \frac{(-1)^{m+2}}{q_{m+2}q_{m+1}} + \frac{(-1)^{m+1}}{q_{m+1}q_m}.$$

By Corollary I, the denominator of the first fraction exceeds that of the second. Hence when m is odd the sum in the last member of the equation is positive, and when m is even the sum is negative.

We now see that any recurring fraction of the type considered in section 201 actually represents a number in the sense of section 70. We have seen that the successive odd convergents continually increase, while the even convergents continually decrease, until the difference between a pair of them is very small. Such sequences of numbers we have seen (section 69 ff.) define real numbers.

204. Limit of error. We are now in a position to state a maximum value for the error made in taking any convergent of a continued fraction for the fraction itself.

Theorem. *The maximum limit of error in taking the nth convergent for the continued fraction is less than* $\frac{1}{q_n q_{n+1}}$.

Since by the theorem of the last section the value of the fraction is between any pair of consecutive convergents, it must differ from either of these convergents by less than they differ from each other, that is, by less than $\frac{1}{q_n q_{n+1}}$.

EXERCISES

Find a convergent that differs by less than .001 from each of the following:

1. $\sqrt{6}$.

Solution.

$$\sqrt{6} = 2 + (\sqrt{6} - 2) = 2 + \frac{6-4}{\sqrt{6}+2} = 2 + \frac{1}{\frac{\sqrt{6}+2}{2}}$$

$$= 2 + \frac{1}{2+}\left(\frac{\sqrt{6}+2}{2} - 2\right) = 2 + \frac{1}{2+}\,\frac{\sqrt{6}-2}{2}$$

$$= 2 + \frac{1}{2+}\,\frac{6-4}{2(\sqrt{6}+2)} = 2 + \frac{1}{2+}\,\frac{1}{\sqrt{6}+2} = 2 + \frac{1}{2+}\,\frac{1}{4+}(\sqrt{6}-2).$$

Since the last surd repeats the one in the first equation, we have

$$\sqrt{6} = 2 + \frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{4+}\cdots.$$

$$\frac{p_1}{q_1} = \frac{2}{1};\ \frac{p_2}{q_2} = \frac{5}{2};\ \frac{p_3}{q_3} = \frac{4\cdot 5+2}{4\cdot 2+1} = \frac{22}{9};$$

$$\frac{p_4}{q_4} = \frac{2\cdot 22+5}{2\cdot 9+2} = \frac{49}{20};\ \frac{p_5}{q_5} = \frac{4\cdot 49+22}{4\cdot 20+9} = \frac{218}{89}.$$

Since
$$\frac{1}{q_4 q_5} = \frac{1}{20\cdot 89} = \frac{1}{1780} < .001,$$

we see by section 204 that $\frac{49}{20}$ satisfies the condition of the problem.

2. $\sqrt{7}$.
3. $\sqrt{46}$.
4. $\sqrt{3}$.
5. $\sqrt{19}$.
6. $\sqrt{35}$.
7. $\sqrt{32}$.
8. $\sqrt{61}$.
9. $\sqrt{55}$.
10. $3 + \sqrt{2}$.
11. $\sqrt{99}$.
12. $\sqrt{41}$.
13. $\sqrt{13}$.

CHAPTER XXII

SCALES OF NOTATION

205. General statement. The ordinary numbers with which we are acquainted are expressed in the decimal notation by means of powers of 10. Thus

$$263 = 2 \cdot 10^2 + 6 \cdot 10 + 3.$$

This is the common **scale of notation**, and 10 is called the **radix** of the scale.

In a similar manner, a number might be expressed in any scale with any radix other than 10. If we take 6 as the radix, we shall have in this scale, for instance,

$$207 = 5 \cdot 6^2 + 4 \cdot 6 + 3 = 543,$$

while in radix 10 $\quad 207 = 2 \cdot 10^2 + 0 \cdot 10 + 7.$

In this scale we need only 0 and five digits to express every positive integer.

The fact that the decimal notation has been adopted generally by civilized peoples may depend indirectly on the fact that we have ten fingers and ten toes, thus affording in prehistoric times a convenient unit in terms of which to count.

Every number whose last digit is zero expressed in the decimal notation is divisible by each of the two factors 2 and 5 which are contained in the radix. If it were possible to establish a system from a fresh start, it is a fair question whether a system with the radix 12 would not be more convenient. At any rate each number expressed with the radix 12 whose last digit is zero would be divisible by 2, by 3, by 4, and by 6. This would reduce by about half the percentage of numbers that would require the fractional notation.

In general, if r is the radix of a scale of notation, any positive integer N will be denoted in this scale as follows:

$$N = a_0r^n + a_1r^{n-1} + a_2r^{n-2} + \cdots + a_n, \qquad (1)$$

where the a's are integers less than r.

One observes that the symbol 10 indicates the radix in any system. In this general scale we need a 0 and $r-1$ digits to express every possible number.

206. Change of scale. If we have a number in the scale of radix r, we may find the expression for that number in the common scale by writing the number in form (1), section 205, and carrying out the indicated operations.

EXAMPLE

Convert 3721, where $r = 4$, into the ordinary scale.

Solution.
$$\begin{aligned} 3721 &= 3 \cdot 4^3 + 7 \cdot 4^2 + 2 \cdot 4 + 1 \\ &= 3 \cdot 64 + 7 \cdot 16 + 8 + 1 \\ &= 313. \end{aligned}$$

If we have a number in the common scale, we may express it in the scale with radix r as follows:

If the number is N, we have to determine the integers $a_0, a_1, a_2, \cdots, a_n$ in the expression

$$N = a_0r^n + a_1r^{n-1} + \cdots + a_n. \qquad (1)$$

Divide (1) by r. We have

$$\frac{N}{r} = a_0r^{n-1} + a_1r^{n-2} + \cdots + a_{n-1} + \frac{a_n}{r} = N' + \frac{a_n}{r};$$

that is, the remainder a_n of this division is the last digit in the expression desired.

Divide N' by r and we obtain

$$\frac{N'}{r} = a_0r^{n-2} + a_1r^{n-3} + \cdots + \frac{a_{n-1}}{r};$$

that is, the remainder from this division is the next to the last digit in the desired expression. Proceeding in this way we obtain all the digits $a_n, a_{n-1}, \cdots, a_1, a_0$.

EXAMPLE

Express 25,395 in the scale with radix 8.

Solution.

$$\begin{array}{r|ll} 8 & 25395 & \\ 8 & 3174 & \text{remainder } 3 \\ 8 & 396 & \text{remainder } 6 \\ 8 & 49 & \text{remainder } 4 \\ 8 & 6 & \text{remainder } 1 \\ & 0 & \text{remainder } 6 \end{array}$$

The number in scale $r = 8$ is 61,463.

To change a number from any scale r_1 to any other scale r_2, we may first change the number to the scale of 10, and then, by the process just given, change from the scale of 10 to the scale r_2.

EXAMPLE

Change 59,043 from scale $r = 4$ to one in which $r = 6$.

Solution. $59{,}043 = 5 \cdot 4^4 + 9 \cdot 4^3 + 0 \cdot 4^2 + 4 \cdot 4 + 3$

$= 1875$ in scale of 10.

Changing from the scale of 10 to the scale of 6,

$$\begin{array}{r|ll} 6 & 1875 & \\ 6 & 312 & \text{remainder } 3 \\ 6 & 52 & \text{remainder } 0 \\ 6 & 8 & \text{remainder } 4 \\ 6 & 1 & \text{remainder } 2 \\ & 0 & \text{remainder } 1 \end{array}$$

Thus 59,043 in a scale of 4 becomes 12,403 in a scale of 6.

207. Fundamental operations. In the four fundamental operations in the common scale we carry and borrow 10 in computing. In computing in a scale of radix 6, for instance, we should carry and borrow 6. If the radix were r, we should carry and borrow r.

Thus let $r = 5$. Then $8 + 3 = 2 \cdot 5 + 1 = 21$. Similarly, $2 \cdot 6 = 2 \cdot 5 + 2 = 22$. This is precisely analogous to our computation in the common scale, where, for instance, we should have $7 + 5 = 1 \cdot 10 + 2 = 12$, or $3 \cdot 6 = 1 \cdot 10 + 8 = 18$.

EXERCISES

Perform the following operations:

1. $5443 + 2053 + 1452$; $r = 6$.

Solution.

$$\begin{array}{r} 5443 \\ 2053 \\ 1452 \\ \hline 13432 \end{array}$$

In this process, since $3 + 3 + 2 = 8$ and is 2 more than the radix, we write down 2 and carry 1. In the next column, $4 + 5 + 5 + 1 = 2 \cdot 6 + 3$ and we therefore write down 3 and carry 2. The next column gives $4 + 0 + 4 + 2 = 1 \cdot 6 + 4$, so we write 4 and carry 1. The last column is $5 + 2 + 1 + 1 = 1 \cdot 6 + 3$, and we write 13.

2. $6402 - 5421$; $r = 7$.

Solution.

$$\begin{array}{r} 6402 \\ 5421 \\ \hline 651 \end{array}$$

In the first column 1 from 2 is, of course, 1. In the next column the zero represents the radix 7, provided we borrow 1 from the adjacent 4. Thus 2 from the radix 7 leaves 5. In the next column we must take 4 from the 3 left after the last subtraction. This we can do only by borrowing 1 from the 6 in the last column. We must then subtract 4 from $7 + 3$, which leaves 6.

3. $7462 \cdot 253$; $r = 8$.

Solution.

$$\begin{array}{r} 7462 \\ 253 \\ \hline 26626 \\ 45772 \\ 17144 \\ \hline 2423146 \end{array}$$

In multiplying by 3 we have nothing to carry until we multiply 6 by 3. Then we get $2 \cdot 8 + 2$, so we write 2 and carry 2. The addition of the partial products is carried out as in Exercise 1.

4. $224{,}332 \div 32$; $r = 5$.

Solution.

$$\begin{array}{r|r|l} 32 & 224332 & 3401 \\ & \underline{201} & \\ & 233 & \\ & \underline{233} & \\ & 32 & \\ & \underline{32} & \\ & 00 & \end{array}$$

In estimating the first place in the quotient we divide 22 by 3, taking care to remember that $22 = 2 \cdot 5 + 2$. Thus 3 is contained in 12 exactly 4 times; but since the complete divisor is 32, we are compelled to use 3 as the first figure of the quotient. The multiplications and subtractions are performed as shown in previous exercises.

5. $7563 + 5621 + 4371$; $r = 8$.

6. $64,523 - 35,426$; $r = 7$.

7. $4132 \cdot 1342$; $r = 5$.

8. $2,471,606 \div 287$; $r = 9$.

9. $756,321 - 647,342$; $r = 8$.

10. $64,531 \cdot 2054$; $r = 7$.

11. $231,423 \cdot 1333$; $r = 5$.

12. $6,548,764 \cdot 23,548$; $r = 9$.

13. $21,033 + 32,111 + 20,021 + 3,102,012$; $r = 4$.

14. $34,052 + 1032 + 35,202 + 55,441$; $r = 6$.

15. $40,043,432 \div 442$; $r = 5$.

208. Fractions. In the ordinary notation we express fractional numbers by digits following the decimal point. This notation may also be used in a scale with any radix.

Thus the expression .3264 stands for

$$\frac{3}{10} + \frac{2}{10^2} + \frac{6}{10^3} + \frac{4}{10^4}$$

in the common scale.

In the scale with radix r it stands for

$$\frac{3}{r} + \frac{2}{r^2} + \frac{6}{r^3} + \frac{4}{r^4}.$$

The process of changing the scale for fractions is performed in accordance with the same principles as are employed in the change of scale for integers. The following examples suffice to illustrate it.

EXAMPLES

1. Express .3264 in the scale of 7 as a decimal fraction.

Solution.

$$.3264 = \frac{3}{7} + \frac{2}{7^2} + \frac{6}{7^3} + \frac{4}{7^4}$$

$$= \frac{3 \cdot 7^3 + 2 \cdot 7^2 + 6 \cdot 7 + 4}{7^4} = \frac{1173}{2401} = .4885 \cdots.$$

2. Express .549 as a fraction for radix 7.

Solution. $$.549 = \frac{a}{7} + \frac{b}{7^2} + \frac{c}{7^3} + \frac{d}{7^4} + \cdots.$$

Multiply by 7: $3.843 = a + \frac{b}{7} + \frac{c}{7^2} + \frac{d}{7^3} + \cdots.$

Thus $a = 3$, and $.843 = \frac{b}{7} + \frac{c}{7^2} + \frac{d}{7^3} + \cdots.$

Multiply by 7: $5.901 = b + \frac{c}{7} + \frac{d}{7^2} + \cdots.$

Thus $b = 5$, and $.901 = \frac{c}{7} + \frac{d}{7^2} + \cdots.$

Multiply by 7: $6.307 = c + \frac{d}{7} + \cdots.$

Thus $c = 6$, and $.307 = \frac{d}{7} + \cdots.$

Multiply by 7: $2.149 = d + \cdots.$
Thus $d = 2$.
The fraction in radix 7 is $.3562 \cdots.$

EXERCISES

Express the following as decimal fractions:

1. $.537$; $r = 8$.
2. $.4463$; $r = 7$.
3. $.2031$; $r = 5$.
4. $.3004$; $r = 6$.

5. Express the decimal fraction .386 as a radix fraction for $r = 9$.

6. Express the decimal fraction .30214 as a radix fraction for $r = 5$.

7. Express the decimal fraction .33333 as a radix fraction for $r = 4$.

8. Express the decimal fraction .79054 as a radix fraction for $r = 7$.

9. In what scale is 72 expressed as 2200?

Solution. We seek r so that
$$2\,r^3 + 2\,r^2 = 72.$$

By synthetic division,
$$\begin{array}{rrrr|l} 2 & +2 & +0 & -72 & \underline{2} \\ & 4 & +12 & +24 & \\ \hline 2 & +6 & +12 & -48 & \end{array}$$

$$\begin{array}{rrrr|l} 2 & +2 & +0 & -72 & \underline{3} \\ & +6 & +24 & +72 & \\ \hline 2 & +8 & +24 & & \end{array}$$

Thus 3 is the value sought.

Check. $2 \cdot 3^3 + 2 \cdot 3^2 = 54 + 18 = 72.$

10. In what scale is 125 denoted by 236?

11. In what scale is 1225 denoted by 1,200,101?

12. In what scale is 3247 denoted by 100,442?

13. Show that 2197 is a perfect cube.

HINT. In what scale is this number represented by 1000?

209. Duodecimals. We may apply some of the foregoing processes to mensuration. If we take one foot as a unit and the radix as 12, we may express distances in the so-called duodecimal notation. Thus 2 feet 6 inches is represented in the duodecimal scale by 2.6. Since in the scale of radix r we need $r - 1$ symbols, we will let $10 = t$ and $11 = e$. Thus 21 feet 10 inches would be expressed in duodecimal notation as $19.t$. We may now find areas and volumes in this notation much more readily than by the usual method of converting all distances to inches.

EXAMPLE

Multiply 5 ft. 7 in. by 8 ft. 4 in.

Solution. We multiply 5.7 by 8.4 in the duodecimal scale.

$$\begin{array}{r} 5.7 \\ \underline{8.4} \\ 1t4 \\ \underline{388} \\ 3t.64 \end{array}$$

To convert the result to square feet and square inches we must keep in mind that $3t.64 = 3 \cdot 12 + 10 + \frac{6}{12} + \frac{4}{144} = 46$ square feet 76 square inches, since 144 square inches equal one square foot.

This example leads to the following method of multiplying distances:

Rule. Express the distances in duodecimal notation with the foot as a unit.

Multiply in the scale for which $r = 12$.

In the product change the part on the left of the point from duodecimal to decimal scale.

Multiply the digit following the point by 12, and add to the last figure to obtain the square inches in the result.

EXERCISES

Multiply the following:

1. 12 ft. 11 in. by 87 ft. 10 in.

Solution.

$$\begin{array}{r}
10.e\\
\underline{73.t}\\
t92\\
329\\
\underline{765}\\
7t6.62
\end{array} = 1134 \text{ sq. ft. } 74 \text{ sq. in.}$$

2. 10 ft. 4 in. by 13 ft. 8 in.

3. 6 ft. 3 in. by 15 ft. 2 in.

4. 75 ft. 11 in. by 23 ft. 10 in.

5. 38 ft. 7 in. by 24 ft. 9 in.

6. What is the area of a room 22 ft. 6 in. by 15 ft. 8 in?

7. What is the area of a window 5 ft. 6 in. by 3 ft. 4 in.?

8. What is the area of a triangular park whose side is 150 feet 4 inches long and whose altitude on that side is 80 feet 6 inches?

HINT. Area $= \frac{1}{2} ab$.

INDEX

Abscissa, 55
Addition, 1; associative law of, 7; commutative law of, 7; of complex numbers, 179; of fractions, 5, 35; of imaginary numbers, 176; of polynomials, 9; of radicals, 76; of zero, 3
Antilogarithms, 102
Applications of variation, 43
Approximation, of irrational numbers, 72; of irrational roots, 216
Arithmetic means, 164
Arithmetic progression, 163; *n*th term of, 163; sum of, 164
Associative law, of addition, 7; of multiplication, 8
Axis of graph, 55

Binomial, square of a, 14
Binomial theorem, 157
Binomials, 26
Briggs logarithms, 94

Change of scale of notation, 291
Change of sign, 204
Characteristic of logarithms, 94
Circular permutations, 229
Coefficients and roots, 195
Columns and rows, identical, 247; interchange of, 246
Combinations, 222, 226
Common difference, 163
Common factor, highest, 27
Common logarithms, 94
Common multiple, least, 30
Commutative law, of addition, 7; of multiplication, 8
Completing the square, 116
Complex numbers, 175, 178; addition and subtraction of, 179; conjugate, 181; division of, 181; equality of, 179; graphical representation of, 178, 180; multiplication of, 181
Compound interest, 106
Condition, equations of, 46
Conic sections, 142
Conjugate complex numbers, 181
Constant, multiplication of determinant by, 245; multiplication of roots by, 200
Continuation of sign, 204
Continued fractions, 277; convergents of, 279; expressing a surd, 284; infinite, 277; recurring, 281; terminating, 277
Continuity of graph, 210
Convergents, 279; properties of, 286
Coördinates, 56

Degeneration of quadratic equations, 135
Dependent equations and systems, 58
Descartes' rule of signs, 204; proof of, 205
Determinants, 236; development of, 237, 242; evaluating, 254; multiplication by a constant, 245; number of terms of, 245; sum of, 253; vanishing of, 253
Development of the determinant, 237, 242; by minors, 249; proof of, 251
Development of partial fractions, 266, 271
Diagonal, principal, 237
Difference, 2; common, 163; of roots, 129
Difference of squares, 22; reduction to, 23
Diminishing roots, 212; graphical interpretation of, 214
Direct variation, 41
Discriminant, 127
Distributive law, 8
Division, 4; of complex numbers, 181; of fractions, 5, 35; of imaginaries, 177; of monomials, 15;

of polynomials, 16; of radicals, 77; synthetic, 186; by zero, 6
Duodecimal scale, 296

Elements of determinant, 237
Ellipse, 141
Equality of complex numbers, 179
Equations, 46; degeneration of, 135; dependent, 58; diminishing roots of, 212; exponential, 106; formation of, 197; graph of, 131, 188; homogeneous, 147, 152, 262; incompatible, 60; involving radicals, 81; linear, 47, 236, 257; of *n*th degree, 184; quadratic, 115, 125; roots of, 190; simultaneous, 58, 138; solution of, 46; theory of, 184
Equivalent equations, 58
Equivalent fractions, 33
Error, limit of, 289
Evaluating determinants, 254
Exponential equations, 106
Exponents, negative, 84; fractional, 84
Expressions, integral and rational, 8
Extraneous solutions, 81

Factor, rationalizing, 79
Factoring, 19; of binomials, 26; by grouping terms, 21; solution of equations by, 115
Factors, 19; highest common, 27; monomial, 20
Form, quadratic, 119
Form of graph of quadratic equations, 133
Formation of equations, 197
Formula, solution by, 118
Formulas, 41, 43, 105, 106, 110, 118
Fractional exponents, 84
Fractional numbers, 6
Fractions, 32; addition of, 35; continued, 277; division of, 35; equivalent, 33; fundamental operations on, 5; multiplication of, 35; reduction of, 32; subtraction of, 35; with various scales of notation, 294
Function, notation for, 184
Fundamental operations, 1, 6; with different scales of notation, 292; on fractions, 5

Generalized powers, 89
Geometric means, 169
Geometric progression, 168; *n*th term of, 169; sum of, 169
Graphical interpretation, of diminishing roots, 214; of roots, 192, 194
Graphical solution, 55, 57
Graphs, of complex numbers, 178; continuity of, 210; of equations, 125, 131, 132, 133, 140, 142, 188; of homogeneous equations, 152
Grouping terms, factoring by, 21

Highest common factor, 27
Homogeneous equations, 147, 152, 262
Horner's method, 216
Hyperbola, 142

Identical rows and columns, 247
Identities, 46
Imaginary numbers, 175; addition and subtraction of, 176; multiplication and division of, 177
Imaginary roots, 193, 194
Incompatible equations, 60
Independent equations, 58
Index of radical, 74
Indices, theory of, 84
Induction, mathematical, 154
Infinite continued fractions, 277
Infinite series, 172; sum of, 173
Integers, 1
Integral expressions, 8
Integral numbers, 6
Integral roots, 207
Interchange of rows and columns, 246
Interest, compound, 106
Interpolation, 100, 101
Inverse variation, 41
Inversion, 242
Irrational numbers, 68, 74; approximation of, 72; notation for, 73; operations on, 72
Irrational roots, 209; approximation of, 216

Joint variation, 42

Law, associative, of addition, 7; associative, of multiplication, 8; commu-

tative, of addition, 7; commutative, of multiplication, 8; distributive, 8
Laws of operation, 7
Least common multiple, 30, 33
Limit of error, 289
Linear equations, 47, 55, 66, 236, 257; homogeneous, 262
Location principle, 210
Logarithms, 89, 90; Briggs system, 94; characteristic of, 94; mantissa of, 94; operations on, 92; tables of, 98, 99; of trigonometric functions, 111, 112

Mantissa, 94
Mathematical induction, 154
Maxima, 134
Mean proportion, 40
Means, arithmetic, 164; geometric, 169
Minima, 134
Minors, development by, 249; proof of, 251
Minuend, 2
Monomial, division of, 15; multiplication of, 12
Monomial factors, 20
Multiple, least common, 30
Multiplication, 4; of complex numbers, 181; of determinants by a constant, 245; of fractions, 5, 35; of imaginaries, 177; of monomials, 12; of polynomials, 13; of radicals, 77; of roots by a constant, 200; types of, 14

Nature of roots of quadratic equations, 127
Negative exponents, 84
Negative numbers, 6
Negative roots, 206
Notation, for irrational numbers, 73; scales of, 290
Number of roots of equations, 126
Number of solutions, 144
Numbers, complex, 175, 178; fractional, 6; integral, 6; irrational, 68; negative, 6; rational, 6; square root of, 70

Operation, laws of, 7
Operations, fundamental, 1; on polynomials, 8
Ordinate, 55

Pair of equations, solution of, 57
Parentheses, 10
Partial fractions, 266; development of, 266, 271, 273; general case, 274
Permutations, 222, 224; circular, 229; involving duplicates, 230
Plotting quadratic equations, 140
Polynomials, addition of, 9; division of, 16; highest common factor of, 27; multiplication of, 13; operations on, 8; radical, 88; square root of, 69; subtraction of, 9
Powers, generalized, 89
Prime factors, 19
Principal diagonal, 237
Principal roots, 71, 85
Probability, 222, 232; of successive choices, 234
Problems, involving quadratics, 121; solution of, 51, 63, 121
Product of roots, 129
Progression, arithmetic, 163; geometric, 168
Proof of development by minors, 251
Properties, of convergents, 286; of quadratic equations, 125
Proportion, 39
Pure quadratic, 115

Quadratic equations, 115; degeneration of, 135; graph of, 131, 132, 133, 142; number of roots of, 126; problems involving, 121; properties and graphs of, 125; simultaneous, 138, 142
Quadratic form, 119
Quadratic trinomial, 24; factors of, 24

Radical polynomials, 88
Radicals, 68; addition and subtraction of, 76; equations involving, 81; index of, 74; multiplication and division of, 77; reduction of, to simplest form, 75
Radix, 290
Ratio, 39; of geometric progression, 168
Rational expressions, 8
Rational numbers, 6
Rational roots, 207, 208
Rationalization, 79

Rationalizing factor, 79
Reciprocal, 35
Recurring continued fractions, 281
Reduction, of fractions, 32; of radicals, 75
Remainder theorem, 185
Root, square, 69, 70; principal, 71, 85
Roots, approximation of, 216; and coefficients, 195; diminishing, 212, 214; of equations, 126, 127, 129, 190; extraneous, 81; graphical interpretation of, 192; imaginary, 193; integral, 207; irrational, 209; multiplication of, by constant, 200; negative, 206; rational, 207, 208
Rows and columns, interchange of, 246; identical, 247
Rule of signs, Descartes's, 204

Scales of notation, 290; change of, 291; duodecimal, 296; fractions with various, 294; fundamental operations with various, 292
Sequences expressing irrational numbers, 72
Series, arithmetic, 163; geometric, 168; infinite, 172
Sign, change of, 204; continuation of, 204
Signs, Descartes's rule of, 204
Simultaneous equations, 58, 59, 66, 138, 236
Solution, by completing the square, 116; by determinants, 236, 257; of equations, 46, 66; by factoring, 115; by formula, 118; graphical, 55, 59; of problems, 51, 63; by substitution, 138; by trial, 196
Solutions, extraneous, 81; number of, 144
Square, of a binomial, 14; completing the, 116; perfect, 23
Square root, of polynomials, 69; of numbers, 70
Squares, difference of, 22, 23
Substitution, solution by, 138
Subtraction, 1; of complex numbers, 179; of fractions, 5, 35; of imaginary numbers, 176; of polynomials, 9; of radicals, 76; of zero, 3
Subtrahend, 2
Sum, 1; of determinants, 253; of infinite series, 173; of roots, 129; of series, 164, 169
Surd as a continued fraction, 284
Synthetic division, 186
Systems, dependent, 58; whose determinants are zero, 259

Tables, of logarithms, 98, 99; of values, 189
Tabular difference, 101
Terminating continued fractions, 277
Terms of determinant, 245
Theorem, binomial, 157; remainder, 185
Theory of equations, 184
Trial, solution by, 196
Trigonometric functions, 111, 112
Trigonometry, 110
Trinomial, quadratic, 24
Trinomial squares, 23
Two unknowns, equations in, 55
Types of multiplication, 14

Unknown, 47

Values, table of, 189
Vanishing of determinant, 253
Variable, 47
Variation, 39; applications, 43; direct, 41; inverse, 41; joint, 42

Zero, 3; division by, 6
Zero determinant, 259

195

128

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